national program close to the center of this region and thus afford our members in the South a good opportunity to attend a national meeting and to meet teachers personally from the other parts of the country. This lack of attendance is in part to be explained, doubtless, by the financial status of the country at the present time. Many of those present spoke of the great value of the meetings and all found pleasure in making new personal contacts.

The sessions of the American Association for the Advancement of Science began on Monday evening with an address in the Municipal Auditorium by Doctor R. P. Strong of Harvard University in the absence through sickness of the retiring president, Doctor Thomas Hunt Morgan, and of the president for the year, Doctor Franz Boas. The lecture was followed by a general reception given by officers of the American Association and the officers of the general local committee. Lectures of general interest were given by prominent scientists each evening during the week; these included an Edison memorial program on Wednesday evening and an address on Friday evening by Doctor Irving Fisher of Yale University on "First principles of booms and depressions."

The Council of the American Association met Monday afternoon and each morning thereafter for the transaction of general business. Professor H. H. Mitchell of the University of Pennsylvania was elected vice-president and chairman of Section A for the year 1932. Professor D. R. Curtiss of Northwestern University was re-elected a member of the Executive Committee.

The mathematicians were comfortably housed in one of the dormitories of Sophie Newcomb College, and had their meals together in the same building. It was a very unusual and delightful experience to have southern cooking, including southern coffee. Many of the visitors made their way to the famous restaurants in the French Quarter for some of their meals. At the lunch on Monday, President Dinwiddie of Tulane University was present and gave a hearty welcome to the mathematics group. A resolution was adopted by rising vote at the Thursday afternoon session of the Association expressing to the local committee of the Tulane and Sophie Newcomb faculty the appreciation of the members for the great hospitality shown, and the ample provision made to insure the comfort and convenience of the visitors.

The annual dinner was held at the Patio Royale in the French Quarter with one hundred sixty-five present. Under the toastmastership of President Eisenhart of the Society, Professor Buchanan welcomed the guests and expressed the hope, certainly realized in high degree, that the week was proving agreeable; he spoke in recognition of the value of the meetings of the two organizations to the mathematics teachers of the South. Professor Slaught described as a dream a picture of the celebration in January 1966 of the fiftieth anniversary of the founding of the Mathematical Association, with Professors Hedrick, Cairns, Slaught, Finkel and many others present and in the full bloom of their years, as at the time of the organization at the beginning of 1916. He prophesied the future growth and prosperity of the publications and other activities of the Society and Association in these fifty years, and spoke also of the fine spirit which

is continually existing in the relations of these organizations in their joint work over the country. After brief remarks by Professor Latimer, the chairman of the program committee, and by Professor Radó, who was introduced as a representative of those born abroad who have adopted this as their country, Professor Jackson, retiring as an editor of the Transactions, described the financial stress which the Society has undergone the past few years as a not unmixed evil. We should have, he said, a collective sense that the work we are doing is a part of the whole work about us, that the work which will be read with interest by a few persons may be more inspiring than work that might appeal to many more. We should learn to estimate not so much the work of others as the work of ourselves and its possible value to others. The value of research is to the man who does the research, in relation to his college or university. Everyone should do so much independent investigation of mathematical ideas that no one person should expect to publish everything that he finds out, but pruning and evaluating he should then present his results to the mathematical journals.

By vote of the members present at the dinner, the Secretaries were instructed to send a telegram of greeting to Professor J. W. Young with an expression of hope for an early recovery from the illness which occasioned his resignation as chairman of the Joint Committee on Funds of the two organizations.

The American Mathematical Society held sessions on Monday morning and afternoon and Wednesday morning for the reading of papers. On Tuesday morning the Society had a joint session with the American Physical Society at which Professor G. D. Birkhoff of Harvard University spoke on "Stability and instability of physical systems," and Doctor W. F. G. Swann of the Bartol Research Foundation gave an address on "The significance of the fundamental concepts of modern atomic theories"; abstracts of these two addresses appeared in *Science* for February 5. At the joint session of the Society with the Physical Society and Section A on Tuesday afternoon, Professor G. A. Bliss gave his retiring address as the Vice-President of Section A, his subject being "The calculus of variations of the quantum theory"; this address will appear in the *Bulletin* of the Society. The Ninth Josiah Willard Gibbs lecture on "Statistical mechanics and the second law of thermodynamics" was given by Professor P. W. Bridgman of Harvard University later Tuesday afternoon. A short abstract of this lecture will be found in *Science* and the paper is to appear in the *Bulletin* of the Society.

At the close of the Wednesday morning program of the Society, the Cole Prize in the Theory of Numbers was presented to Professor H. S. Vandiver of the University of Texas for his work on Fermat's Last Theorem. By invitation he gave a resumé of the advances which he himself has made toward the solution of this problem.

The program of the Mathematical Association consisted of a joint session with the Society on Thursday morning and a separate session on Thursday afternoon, with President E. T. Bell in the chair. A good program was provided by the committee consisting of Professors Julia Dale, H. L. Smith and C. G.

Latimer, chairman. The program follows, together with abstracts of some of the papers, numbered in accordance with their place on the program.

JOINT SESSION OF THE ASSOCIATION WITH THE SOCIETY

1. "Non-analytic functions of a complex variable" by Professor E. R. HEDRICK, University of California at Los Angeles, retiring president of the Society.

2. "Some recent researches in the theory of numbers" by Professor R. D. CARMICHAEL, University of Illinois, representing the Association.

1. The address by Professor Hedrick will appear in an early issue of the *Bulletin* of the Society.

2. The address by Professor Carmichael appears in this issue of the MONTHLY.

SEPARATE SESSION OF THE ASSOCIATION

1. "Maximum efficiency in freshman mathematics" by Professor C. R. SHERER, Texas Christian University.

2. "A method of handling certain elementary integrals" by Professor OTTO DUNKEL, Washington University.

3. "Fourier series" by Professor DUNHAM JACKSON, University of Minnesota.

1. The average high school graduate has been thoroughly steeped in the idea of avoiding anything that looks or sounds mathematical. Under the influence of the aim of practicality, reorganizations of curricula have changed mathematics, both in its position in the curricula and with regard to course content. The need for serious consideration of the situation with a view to retaining mathematics as one of the essential tool courses, and even as a cultural subject, is obvious.

The following plan was initiated in Texas Christian University in the fall semester of 1930–31. Its purpose was two-fold:

> 1. To aid those students who were poorly prepared in their high school mathematics or who had forgotten most of the fundamental principles necessary for successful study in college mathematics.
>
> 2. To stimulate and motivate those students of exceptional ability and capacity.

Three sections of freshman mathematics are scheduled for the same hour; for example, eight o'clock Monday, Wednesday, and Friday. In order that schedule problems shall be minimized, all students registering for that class cannot register for another course at the same hour on Tuesday, Thursday, and Saturday. The three sections meet together for the first class hour, where the proper psychological atmosphere is very carefully prepared by a staff member who understands and approves the plan. A comprehensive review of elementary algebra is given for a period of three weeks. This is followed by a suitable uniform placement examination.

Guided by the results of this examination, the high school record, the general intelligence tests, and, in doubtful cases, by personal interviews, A, B, and C sections are formed. The A and B sections are composed of the exceptional and average students respectively, while the C section is comprised of all those whose ability or preparation is below normal. The A and B sections continue with the normal schedule of three days per week, but the C section meets five times for three hours credit. The content of the course for the C section is so constructed that an ambitious student will get a foundation sufficient to continue in the next course.

In order that the superior or exceptional people may advance in proportion to their capacity, a special class is formed for the second semester. They are chosen on the following bases:

1. Record in the first semester's work.
2. Special examination to be taken by all those who wish to become candidates.
3. Recommendation by the faculty.
4. Personal conferences through which aims and attitudes are discovered.

The general class procedure is built upon the seminar plan. Since the membership of the class is small, it is possible to direct a part of their study through individual conferences.

2. This paper presents a method of evaluating the integrals (1) $\int(x^2+2Ax+B)^{-1/2}\,dx$, (2) $\int(x^2+2Ax+B)^{1/2}\,dx$ by use of a simple algebraic substitution and the principle of "composition and division" in elementary algebra. The method appears to involve less computation than the usual methods. This paper appears in this issue of the MONTHLY.

3. Professor Jackson gave an outline of a course in Fourier Series which he has been giving the first semester of the present academic year, adapted to undergraduates who have had but one year of the calculus. This paper made it quite manifest that it is readily possible to carry on such a course for juniors as well as seniors and graduate students. It is to be hoped that Professor Jackson will make this material available for the teachers of undergraduate courses.

MEETINGS OF THE BOARD OF TRUSTEES

Seven members of the outgoing Board of Trustees and eight members of the incoming Board were present at the New Orleans meetings.

The following seventeen persons and three institutions were elected to membership on applications duly certified:

To Individual Membership

ANN S. BUCHANAN, A.M. (La. State Univ.) Instr., Southwestern Louisiana Institute, Lafayette, La.

R. V. CHURCHILL, Ph.D. (Michigan) Asst. Prof., Univ. of Michigan, Ann Arbor, Mich.

H. M. Cox, A.M. (Duke) Instr., Georgia School of Tech., Atlanta, Ga.

A. T. Craig, Ph.D. (Iowa) Asso., Univ. of Iowa, Iowa City, Iowa

J. H. Cross, A.M. (Texas Tech.) Asst., Dept. of Physics, Texas Tech. Coll., Lubbock, Tex.

Sister Francis Xavier, Ph.D. (Fordham) Prof., St. Joseph's Coll. for Women, Brooklyn, N. Y.

F. C. German, A.M. (Illinois) Asso. Prof., State Teachers Coll., Pittsburg, Kans.

Sarah E. Haughton, A.M. (Alabama) Instr., Univ. of Alabama, University, Ala.

R. L. Korgen, A.M. (Harvard) Instr., Bowdoin Coll., Brunswick, Maine.

O. C. Kreider, M.S. (Iowa State Coll.) Teacher, Ellsworth Jr. Coll., Iowa Falls, Iowa

J. E. LaFon, A.M. (Oklahoma) Asst., Univ. of Oklahoma, Norman, Okla.

M. C. Ledesma, Senior, School of Engineering, Milwaukee, Wis.

Sister Charles Mary Morrison, Ph.D. (Catholic Univ.) Prof., Nazareth Coll., Louisville, Ky.

V. F. Murray, M.A. (St. Andrew's Univ., Scotland) Hoboken, N. J.

Viola Perry, A.M. (Peabody) Head of Dept., S. Georgia Teachers Coll., Collegeboro, Ga.

H. C. Searcy, Jr., A.M. (So. Methodist Univ.) Instr., Jr. Coll., Edinburg, Tex.

Catharine L. Walker, A.M. (Columbia) Instr., Jr. Coll., Fort Smith, Ark.

Henry Walther, B.S. (Cooper Union) Research Engr., Bell Telephone Labs., New York, N. Y.

To Institutional Membership

University of New Mexico, Albuquerque, N. M.

Bennett College for Women, Greensboro, N. C.

North Carolina State College of Agriculture and Engineering, Raleigh, N. C.

The financial report of the Secretary-Treasurer for the year 1931 was accepted. Professor Slaught for the finance committee, with Professors C. A. Hutchinson and F. W. Owens, had examined the report and the evidences of assets and declared the report satisfactory.

The Trustees accepted with regret the resignation of Professor J. W. Young as chairman and member of the Joint Committee on Funds, on account of illness. Because the general plans of this committee are necessarily at a standstill at present, it was voted that the chairmanship of the committee remain vacant for the present.

It was voted to approve the following associate editors of the Monthly for the year 1932, as nominated by Professor Carver:

W. F. Cheney	R. E. Gilman	J. R. Musselman
N. A. Court	R. A. Johnson	H. L. Olson
Otto Dunkel	B. W. Jones	D. E. Smith
H. S. Everett	H. W. Kuhn	F. M. Weida
B. F. Finkel		

It was voted to nominate Professor H. L. Rietz as representative of the Association on the National Research Council for a three-year term from June, 11, 1932, in succession to Professor R. D. Carmichael.

It was voted that as soon as practicable the dues of our foreign members should be put on a basis of U. S. exchange.

It was voted to accept with thanks the invitation of Harvard University to

hold the meeting of the Mathematical Association there in connection with the meeting of the American Association in December 1933. It was also voted to hold the meeting of the Mathematical Association at Atlantic City in December 1932 in connection with the meeting there of the American Association.

Some other items of business were transacted which had to do with general and sectional activities.

Annual Business Meeting of the Association

The Secretary announced the names of those who had been elected to membership at the meeting of the Trustees. He reported also the deaths of the following members:

M. Sue Burney, Greenwich, Conn. Formerly instructor of mathematics, Junior College, St. Joseph, Mo. (August 20, 1931).

Annie C. Clark (Mrs. Theron Clark) Head of mathematics dept., Holmby College, Los Angeles, Calif. (January 21, 1931)

G H. Cresse, Professor of mathematics, University of Arizona (May 3, 1931).

S. M. Hadley, Dean and professor of mathematics, Penn College (Retired June, 1930) (November 10, 1931)

Charles Haseman, Professor of mathematics and mechanics, University of Nevada (July 9, 1931)

A. A. Himwich, M.D. New York, N. Y. (April 18, 1931)

H. L. Hodgkins, Dean and professor of mathematics, George Washington University (February 13, 1931)

F. J. Holder, Professor of mathematics and dean of School of Commerce, Mercer University, Macon, Ga. (December 30, 1931)

Alexander Knisely, Columbia City, Ind. (September 29, 1931)

Bessie I. Miller, Formerly professor of mathematics, Rockford College, Ill. (February 4, 1931)

C. L. E. Moore, Associate professor of mathematics, Massachusetts Institute of Technology (December 5, 1931)

G. W. Myers, Professor of mathematics, University of Chicago (November 23, 1931)

G. D. Olds, President emeritus, Amherst College (May 11, 1931)

C. I. Palmer, Professor of mathematics and dean of students, Armour Institute of Technology (April 9, 1931)

C. W. Watts, Professor of mathematics, Virginia Military Institute (July 10, 1931)

W. A. Zehring, Professor of mathematics, Purdue University (May 1, 1931)

The election of officers for the year 1932 resulted in the following, as reported by the tellers, Professors H. M. Hosford and F. M. Weida:

For Vice-Presidents: W. H. Bussey, 307 votes; Abraham Cohen, 259 votes; G. C. Evans, 286 votes; Mayme I. Logsdon, 184 votes.

For additional members of the Board of Trustees, to serve until January 1935: C. R. Adams, 223 votes; W. C. Brenke, 212 votes; H. E. Buchanan, 280 votes; W. B. Ford, 336 votes; E. R. Hedrick, 406 votes; W. R. Longley, 250 votes; W. E. Milne, 178 votes; J. L. Walsh, 186 votes.

The following were accordingly declared elected:

Vice-Presidents: W. H. Bussey, University of Minnesota; G. C. Evans, Rice Institute.

Additional members of the Board of Trustees: H. E. Buchanan, Tulane

University; W. B. FORD, University of Michigan; E. R. HEDRICK, University of California at Los Angeles; W. R. LONGLEY, Yale University.

By action of the Association the Secretary was instructed to write the following letter to the retiring editor-in-chief, Professor W. H. Bussey:

MY DEAR PROFESSOR BUSSEY:

I have the pleasure of informing you that at the session of the Mathematical Association of America on Thursday afternoon, December 31, 1931, the Association members adopted heartily a resolution expressing our appreciation of the notable work which you have done during the past five years as editor-in-chief of the American Mathematical Monthly. They asked me to express to you our recognition of the high ideals that have characterized the Monthly during your term of office and to thank you for the care and the devotion which you have shown in performing so ably the duties of the office.

Yours cordially,
(Signed) W. D. CAIRNS, *Secretary*

REPORT OF THE SECRETARY-TREASURER AS TREASURER, DECEMBER 14, 1931

RECEIPTS

Balance Dec. 15, 1930		$9,236.65
1929 indiv. dues	48.00	
1930 indiv. dues	334.40	
1930 instit. dues	28.50	
1931 indiv. dues	6,800.72	
1931 instit. dues	787.60	
1931 subscriptions	879.09	
Initiation fees	252.00	
Life memberships	125.47	
Advertising	639.00	
Sale *Register*	1.50	
Sale copies of MONTHLY.	85.50	
Sale First Carus Mon.	30.00	
Sale Second Carus Mon.	36.25	
Sale Third Carus Mon.	40.00	
Sale Fourth Carus Mon.	45.00	
Advance sale Fifth Mon.	1.25	
Sale Catalog	.50	
Sale Rhind Papyrus	990.00	
Transfer from Carus Mon. certif.	83.75	
Carrying charges Papyrus	27.80	
Received *Annals* subscriptions	5.00	
Int. Oberlin Savgs. Bk.	99.26	
Int. Peoples Bkg. Co.	83.57	
Int. Liberty Bonds	42.51	
Int. Hardy Fund	120.00	
Int. certifs. of deposit	19.34	
Int. from Genl. Endowment Fund Bonds	392.50	
Int. Carus Fund	50.00	
Int. Chace Fund	132.50	

EXPENDITURES

Publisher's bills (Nov. '30–Oct. '31)		$5,278.71
President's office		44.25
Joint Committee on Funds		342.63
Manager's office		72.22
Editor-in-chief's office 1931		500.73
Editor-in-chief's office 1932		172.38
Committee on Geometry		4.65
Committee on Membership		86.50
Register		167.00
Printing library catalog		161.49
Secretary-Treasurer's office:		
Postage	374.01	
Bond	5.00	
Safety deposit	4.00	
Office supplies	112.01	
Express, tel., etc.	79.25	
Clerical work	1,685.25	
Printing	347.06	
Library expense	146.20	
Ins. back copies of MONTHLY	11.00	
Paid copies MONTHLY.	33.05	
Cleveland meeting	50.00	
Minneapolis meeting	75.00	
Refund instit. dues	4.50	
Refund 1931 subscriptions	7.20	
Refund Mon. purchase	1.25	2,934.78
Annals subvention		375.00
Paid to sections from initiation fees		85.70
Forwarded *Annals* subscriptions		7.50
Paid *Annals* subscriptions		5.00

Int. Chauvenet Fund...	25.00		Paid B. F. Finkel int. Hardy Fund.	120.00
Int. from investment of current funds.....	180.81	12,386.82	Sustaining memb. in American Math. Society.....	100.00
			Carrying charges Papyrus.....	27.77
Total 1931 receipts.....		$21,623.47	Expense acct. Carus Mons.....	225.00
			Transfer to Carus Mon. Fund.....	124.27
			Transfer to Chace Fund.....	977.26
Total expenditures.....		$11,812.84	Total expenditures.....	$11,812.84
Balance to the end of 1931 business.		9,810.63	Cash on hand.....	10.03
			Checking account.....	1,299.44
			Oberlin Savgs. Bk. acct.....	2,556.26
			Peoples Bkg. Co. acct.....	2,105.38
			Liberty Bonds.....	1,000.00
			Iowa Ry. & Light Co. 5% Bonds...	3,000.00
			Certif. of deposit.....	496.24
Received on 1932 business.....		740.47	Part certif. of deposit.....	83.75
Book balance Dec. 14, 1931.....		$10,551.10	Bank balance Dec. 14, 1931.....	$10,551.10

Exhibit of the Funds of the Association

Carus Monograph Fund

Balance Dec. 15, 1930		$5,756.19
Receipts: Sales	$152.50	
Interest	241.77	394.27
		$6,150.46
Expenditures		225.00
		5,925.46
Certificates of deposit	4,925.46	
Cleveland Trust Securities Co. Gold Bond	1,000.00	
Balance Dec. 14, 1931		$5,925.46

Arnold Buffum Chace Fund

Balance Dec. 15, 1930		$2,930.01
Receipts: Sales	990.00	
For carrying charges	27.80	
Interest	174.94	1,192.74
		$4,122.75
Expenditures: For carrying charges		27.77
		$4,094.98
Iowa Rwy. and Light Co. 5% Bond	$1,000.00	
Western United Gas and Elec. Co. Bonds	1,500.00	
Certificates of deposit	1,449.71	
Cash in bank	145.27	
Balance Dec. 14, 1931		$4,094.98

CHAUVENET PRIZE FUND

Balance Dec. 15, 1930		$550.00
Interest	25.00	
Amount set aside for 1931	20.00	45.00
		$595.00
Iowa Rwy. & Light Co. 5% Bond	500.00	
Cash in bank	95.00	
Balance Dec. 14, 1931		$595.00

LIFE MEMBERSHIP FUND

Liability on life memberships Dec. 15, 1930	$446.31
Received on life membership payments	125.47
	$571.78
To be transferred to current funds, surplus	14.79
Liability on life memberships as of Jan. 1, 1932	$556.99

GENERAL ENDOWMENT FUND

Balance Dec. 15, 1930		$8,000.00
Liberty Bond	$1,000.00	
Land Trust Certificate	1,000.00	
Cleveland Trust Investment Co. Gold Bond	1,000.00	
Idaho Power Co. 5% Bonds	2,000.00	
Northwestern Elec. Co. Bonds	3,000.00	
Balance Dec. 14, 1931		$8,000.00

Of the funds on hand, indicated in the first division of this financial report, $95.00 belongs to the Chauvenet Prize Fund; $145.27 belongs to the Arnold Buffum Chace Fund; and $556.99 is held as a Life Membership Fund, representing the liability on life memberships already paid for, as of date Jan. 1, 1932. Aside from these amounts, the Carus Monograph Fund is carried as a separate fund in the form of two certificates of deposit which bear 4%, compounded quarterly, to the amount of $4,925.46 and a Gold Bond of $1,000.00; $1,449.71 of the Arnold Buffum Chace Fund is similarly carried in the form of two certificates of deposit and $2,500.00 in securities; the contribution of Professor Walter B. Ford to the Chauvenet Fund is carried as a 5% Gold Bond; and the sum of $8,000.00 is held in reserve as the General Endowment Fund, in securities as listed above.

When the accounts were closed Dec. 14, 1931, there remained on the total business for 1931 the following items:

BILLS RECEIVABLE (partly estimated)		BILLS PAYABLE (partly estimated)	
1931 individual dues	$200.00	Publisher's bills (Nov. Dec. '31)	$1,220.00
Advertising	100.00	President's office	30.00
		Manager's office	25.00
	$300.00	Editor-in-chief's office	140.00

Other editors' postage	30.00
Printing *Register*	550.78
Secretary-Treasurer's office	300.00
Annals subvention	75.00
Initiation fees due to sections	900.00
Chauvenet Prize Fund	95.00
Chace Fund	145.27
Life Membership Fund	556.99
	$4,068.04

If to the balance on 1931 business shown in the report, $9,810.63, there be added the bills receivable, $300.00, and there be subtracted the estimated bills payable, $4,068.04, there results an estimated final balance on 1931 business of approximately $6,040, which represents the accumulated surplus in current funds. This is a satisfactory showing for the year when compared with the corresponding figure of $5,280 for a year ago. It is however a disquieting feature of our status that more members are in arrears, too often without any word to the secretary-treasurer. It is very much to be hoped that those who find it necessary to postpone the payment of their dues will inform the office as to when payment may be expected, for arrangements may ordinarily be made in such cases so that the members may retain full standing. In the absence of such information the secretary-treasurer needs, even reluctantly, to conform to the rule of the Trustees, according to which members who are more than one year in arrears do not receive the MONTHLY and members who are more than two years in arrears and have not responded to letters from the office of the secretary-treasurer are omitted from the list of members. It is quite preferable to keep in touch with the secretary-treasurer.

W. D. CAIRNS, *Secretary-Treasurer*

THE TWENTIETH MEETING OF THE IOWA SECTION

The twentieth meeting of the Iowa Section of the Mathematical Association of America was held with the Iowa Academy of Science at the Davenport Public Museum and St. Ambrose College, Davenport, Iowa, on May 1 and 2, 1931. The meetings were held in the Assembly Room of Lewis Hall, St. Ambrose College.

The attendance was about forty, including the following seventeen members of the Association: E. W. Chittenden, L. M. Coffin, Julia T. Colpitts, N. B. Conkwright, J. M. Earl, F. M. McGaw, E. E. Moots, J. F. Reilly, H. L. Rietz, B. D. Roberts, E. R. Smith, G. W. Snedecor, C. W. Strom, John Theobald, L. E. Ward, C. W. Wester, Roscoe Woods.

The Section chairman, Professor G. W. Snedecor, presided at both the Friday afternoon and Saturday morning sessions. The Section joined with the Iowa Academy of Science at dinner on Friday evening.

At the business session officers for 1931–1932 were elected as follows: Chair-

man, C. W. Strom, Luther College; Vice-chairman, B. D. Roberts, Parsons College; Secretary-Treasurer, J. F. Reilly, University of Iowa.

The program consisted of nineteen papers, as follows:

1. "Complete systems under certain finite groups" by Professor C. W. Strom, Luther College.
2. "Some properties of a class of polynomials generated in the successive differences of the Poisson function" by Professor E. R. Smith, Iowa State College.
3. "The convergence of a sequence of polynomials" by Professor J. M. Earl, University of Omaha.
4. "Concerning some properties of the distribution of geometric means of large samples" by Dr. A. T. Craig, University of Iowa, by invitation.
5. "Review of solar observations at Alta, Iowa, for the five years, 1926 to 1930" by Mr. D. E. Hadden, Astronomical Observatory, Alta, Iowa, (by title).
6. "A method of solving numerical equations" by Mr. S. A. Corey, Des Moines, presented by Professor D. L. Holl, Iowa State College.
7. "The standard error of the mean of 'matched' samples from a normal universe" by Dr. S. S. Wilks, University of Iowa, by invitation.
8. "A normalized orthogonal system of functions" by Mr. Fred Robertson, Iowa State College, presented by Professor E. S. Allen.
9. "Simpson's rules for the approximate value of an integral" by Professor J. F. Reilly, University of Iowa.
10. "Determination of the Gram-Charlier series as a solution of Pearson's differential equation" by Professor E. R. Smith, Iowa State College.
11. "The standard deviation of a standard regression coefficient" by Professor G. W. Snedecor, Iowa State College.
12. "On the reduction of a principal part of an unsymmetric kernel to a canonical form" by Professor E. W. Chittenden, University of Iowa.
13. "Grades in a certain course in mathematics as a criterion of teaching ability in the preceding course" by Miss Gertrude Cox, Iowa State College, by invitation.
14. "Some noteworthy solar disturbances observed at Alta, Iowa, during the years 1890 to 1930" by Mr. D. E. Hadden, (by title).
15. "On the problem of n-bodies with a special law of force" by Mr. J. J. Hinrichsen, Iowa State College, by invitation.
16. "Transverse oscillations of a clamped beam" by Mr. A. W. Davis, Iowa State College, by invitation.
17. "The scale in certain conformal representations of the earth upon a plane" by Mr. H. T. Hurley, Storm Lake, by invitation.
18. "Some examples illustrating exceptional cases in elementary differential equations" by Dr. N. B. Conkwright, University of Iowa.
19. "A statistical investigation of the relation between planetary and lunar positions and temperatures in Des Moines" by Mr. Larry Page, Des Moines, by invitation.

Abstracts of some of these papers follow:

1. In this paper Professor Strom determined the complete systems of invariant polynomials under the following groups: (1) the cyclic groups, (2) the solvable groups, (3) the simple group of degree 7 and order 168.

2. The function $\psi(x) = e^{-\lambda}\lambda^x/x!$, defined for positive integral values of x, has its nth difference of the form $P_n(x)\psi(x)$, where

$$P_n(x) = 1 - \frac{nx}{\lambda} + \frac{n(n-1)}{1.2}\frac{x(x-1)}{\lambda^2} + \cdots + (-1)^n\frac{x(x-1)\cdots(x-n+1)}{\lambda^n}.$$

Professor Smith shows that $\sum\psi_n(x)P_m(x) = 0$ when $m \neq n$, and $= m!/\lambda^m$ when $m = n$, and because of this biorthogonal property the expansion of certain functions, such as the frequency function, in terms of a series of these polynomials may be readily obtained.

3. Professor Earl in his paper was concerned with a generalization of expansions in terms of linear combinations of the classical trigonometric polynomials.

5. In his first paper Mr. Hadden reviewed the records he had made at Alta of sunspot activity for the five year period ending 1930, fixing the date of maximum activity as September, 1928.

6. In this paper Mr. Corey considers numerical solutions of $f(x) = 0$, in which the function is algebraic or transcendental, by iteration methods. By writing the equation in the form $f(x) + g(x) - yg(x) = 0$ the roots are expressed as a power series in y by means of a MacLaurin's development. When an approximate value of a root is known a more accurate value may be readily obtained by making a proper choice of $g(x)$. The method may be used to locate both real and imaginary roots.

8. The purpose of Mr. Robertson's paper was to form a normalized orthogonal system of functions over the range zero to infinity for the system of functions

$$Q_n(x,z) = \left[1 - \left\{1 + xz + \frac{x^2z^2}{2!} + \cdots + \frac{x^{n-1}z^{n-1}}{(n-1)!}\right\}e^{-xz}\right]z^{-n}.$$

9. In the derivation of Simpson's rules for the approximate value of an integral Professor Reilly suggested the use of finite rather than infinite expansions, thereby obtaining a reliable, if somewhat indefinite limit for the error.

10. In this paper Professor Smith shows that the solution of the equation $dy/dx = (\alpha+x)y/(\alpha+\beta x+\gamma x^2)$ may be expressed by means of a series of the Gram-Charlier type. A convenient recurrence relation between the successive coefficients makes it possible to express them in terms of the parameters of the differential equation. The method admits of generalization.

15. In his paper Mr. Hinrichsen considers the problem of n bodies with the potential function defined by $\mu = \sum m_i m_j / r_{ij}^d$, $d > 2$. He establishes two theorems concerning real motions of the n bodies, showing that under certain conditions the motion will terminate in a collision, while under other conditions all n bodies separate indefinitely from one another.

16. In Mr. Davis's paper the frequency numbers corresponding to the different modes of the transverse vibrations of a beam are determined by a minimizing process given by W. Ritz. The assumed function in the integral to be minimized satisfies the boundary conditions and contains n parameters, the determinant of which leads to the frequency equations. In the simpler cases these results are compared with the well known accurate results.

17. A combination of expanding scales for vertical and horizontal distances on Mercator's map gives a scale applicable to distances in any direction, the scale of middle scalar distance between the two points being used as a measure of the distance. Mr. Hurley constructed such a scale for the stereographic map, measuring slant distances on the scale of the bounding latitude. These two scales together give the distance between any two points on the earth within a limit of, say, 3000 miles.

18. In his paper Dr. Conkwright discussed singular solutions of certain equations which do not represent envelopes, singular solutions of simultaneous equations, and of total differential equations.

JOHN F. REILLY, *Secretary*

FIFTEENTH ANNUAL MEETING OF THE KENTUCKY SECTION

The fifteenth annual meeting of the Kentucky Section of the Mathematical Association of America was held at the University of Kentucky, Lexington, Kentucky, on Saturday, May 9, 1931. The chairman, Professor J. M. Davis, presided at both morning and afternoon sessions.

There were thirty-eight persons present, including the following seventeen members of the Association: P. P. Boyd, J. M. Davis, H. H. Downing, A. R. Fehn, Lydia K. Fremd, Charles Hatfield, C. G. Latimer, Elizabeth LeStourgeon, Mayme I. Logsdon, C. A. Maney, W. L. Moore, R. S. Park, Sallie Pence, D. W. Pugsley, E. L. Rees, Guy Stevenson, H. M. Yarbrough.

The Section was fortunate in having as its guest Professor Mayme I. Logsdon of the University of Chicago. During the intermission the members and guests had an excellent opportunity to become acquainted in the faculty club room placed at their disposal. A delightful luncheon was served by the University to all present. The following officers were elected for the coming year: Chairman, R. S. Park, Eastern State Teachers College; Secretary, A. R. Fehn, Centre College.

The program of the meeting follows, with abstracts of some of the papers:

1. "Curvature in the Einstein space-time world" by Professor E. L. Rees, University of Kentucky.

2. "The arithmetics of certain generalized quaternion algebras" by Professor J. M. Boswell, Georgetown College, by invitation.

3. "Brief outline of thesis on non-Euclidean geometry" by Miss Alleen Lemons, University of Kentucky, by invitation.

4. "Finite geometries" by Professor W. L. Moore, University of Louisville.

5. "On the class numbers of cubic fields" by Professor C. G. Latimer, University of Kentucky.

6. "Some concepts of mathematical physics" by Professor J. G. Black, Morehead State Teachers College, by invitation.

7. "The slide rule as an aid in teaching mathematics" by Professor D. W. Pugsley, Berea College.

8. "On the place of mathematics in a liberal arts education" by Professor C. A. Maney, Transylvania College.

9. "Reorganization of material for freshman mathematics" by Professor Mayme I. Logsdon, University of Chicago.

1. In this paper the theory of relativity was discussed from the point of view of the curvature of the space-time world and the fundamental importance of this curvature in the theory was emphasized.

4. Dr. Moore's paper is a preliminary report on a method of constructing certain finite plane geometries of $n+1$ points on a line. Let P_{ij} $(i=0, 1, 2, \cdots, n;$ $j=1, 2, \cdots, n)$ and P_{00} be the n^2+n+1 points in the geometry, then the n^2+n+1 lines are selected from the square array as follows:

$$\begin{array}{llll} P_{00} & P_{00} & \cdots & P_{00} \\ P_{01} & P_{11} & \cdots & P_{n1} \\ \cdot & \cdot & \cdots & \cdot \\ P_{0n} & P_{1n} & \cdots & P_{nn}: \end{array}$$

I. The points in each column, giving $n+1$ lines.

II. P_{01} and the points in each row of the square array from which the first row and column have been deleted, giving n lines.

III. P_{0l} $(l=2, 3, \cdots, n)$ and points from each row and column chosen such that no pair occurs in any previous selection, giving $n(n-1)$ lines.

All such geometries can be easily constructed for which there exists a group G_n of order n which is left invariant by a group G_{n-1} of order $n-1$. Those geometries for which the group G_{n-1} is Abelian were found to be Desarguean while those in which the group G_{n-1} is non-Abelian were found to be non-Desarguean. The author has not yet been able to satisfy himself that all finite plane geometries are given by this construction nor that the relation between the character of the group G_{n-1} and the geometry is a general one.

6. Starting with the inverse square formulas for "action at a distance" in the case of electricity, magnetism and gravitation, the merging of electricity and magnetism into the electromagnetic equations of Maxwell was discussed. Mentioning the work of Newton in gravitation and light, the different portions of the light spectrum as known at present were described. The postulate of an ether as a light bearing medium and the very accurate yet unsuccessful attempt to detect absolute motion through it led to the relativity theory which indicated that light had mass and also that mass was convertible into light or radiation.

Millikan's cosmic ray work was reviewed and the analysis of his absorption curves into four components corresponding to the building up of the elements helium, oxygen, silicon and iron according to Einstein's equation for the change of mass into radiation was explained.

7. Professor Pugsley's paper took up briefly the development of the logarithmic slide rule, the question of accuracy, and the usefulness of the slide rule. It was pointed out that the usefulness of the slide rule is three-fold, namely, it can be used to create student interest, it saves time and drudgery in certain computations, and it is very helpful in checking computations by both student and instructor. The discussion was confined to the topic indicated in the title, only brief mention being made of the practical usefulness in other fields.

8. Exhibits were presented by Professor Maney showing that elementary mathematics, especially statistics, is being used increasingly in research in the fields of education, sociology, business administration, and economics. More and more mathematics is being reorganized as the necessary language of scientific thinking. The study of mathematics is therefore a basic part of a liberal education.

A. R. FEHN, *Secretary*

SOME RECENT RESEARCHES IN THE THEORY OF NUMBERS[1]

By R. D. CARMICHAEL, University of Illinois

1. *Introduction.* Recent advances in the theory of numbers have been so numerous and varied that no discussion of all of them in the space of an hour could have that unity which is necessary to the comfort of either the speaker or the hearer. Hence there must be a selection of material. The choice will be determined by the need of procuring unity and by the present interests of the speaker.

Although there have been some recent advances in the theory of integers defined by recurrence relations, and the speaker has himself been interested in this matter, the subject will be omitted for the sake of greater unity. The contributions which have been made in recent years, and particularly in America, to a treatment of Fermat's last theorem will be left entirely out of consideration. No analysis will be given of progress in the theory of algebraic numbers. The most important of all omitted contributions are those of Dickson and his followers. In these researches we have a remarkable characteristic development of the theory of numbers having the closest association with classical points of view and dealing with some of the most interesting questions which have ever arisen in the theory of numbers.

Passing over still other subjects, we shall confine our attention almost en-

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[1] An address delivered by invitation at a meeting of the Mathematical Association of America, at New Orleans, on December 31, 1931.

tirely to positive integers and largely to the asymptotic properties of number-theoretic functions and the expansion of the latter in infinite series. Such problems frequently begin from what seem to be simple ideas but their treatment often requires the most abstruse methods. In the main the problems which we shall examine have this quality. But there is one of them (that treated in §VI) which deals with a question that has not yet passed beyond the most elementary stage. It therefore invites further treatment. In fact, it is of such character that the amateur may well be able to do some work of value, particularly if his results are put into the hands of one trained in the deeper methods of the theory of numbers.

I. THE WORK OF SRINIVASA RAMANUJAN

2. *Srinivasa Ramanujan.* During our generation no more romantic personality than that of Srinivasa Ramanujan has moved across the field of mathematical interest. Indeed it is true that there have been few individuals in human history and in all fields of intellectual endeavor who draw our interest more surely than Ramanujan or who have excited more fully a certain peculiar admiration for their genius and their achievements under adverse conditions. There is nothing particularly noteworthy about Ramanujan's ancestry to account for his great gifts. His father and paternal grandfather were petty accountants in Kumbakonam, an important town in the Tanjore district in India. His mother was a woman of strong common sense. For some time after marriage she was without children. Her father prayed to a famous local goddess to bless his daughter with children; and shortly afterwards Ramanujan, her eldest child, was born.

I can not pause to speak of his meager and somewhat irregular school and college education. On January 16, 1913, at 25 years of age, he wrote to G. H. Hardy a letter containing the following words:

> I have had no University education but I have undergone the ordinary school course. After leaving school I have been employing the spare time at my disposal to work at Mathematics. I have not trodden through the conventional regular course which is followed in a University course, but I am striking out a new path for myself.

On February 27 of the same year he wrote to Hardy as follows:

> I have found a friend in you who views my labours sympathetically. This is already some encouragement to me to proceed. . . . I find in many a place in your letter rigorous proofs are required and you ask me to communicate the methods of proof. . . . I told him that the sum of an infinite number of terms of the series $1+2+3+4+\cdots=-1/12$ under my theory. If I tell you this you will at once point out to me the lunatic asylum as my goal. . . . What I tell you is this. Verify the results I give and if they agree with your results . . . you should at least grant that there may be some truths in my fundamental basis. . . .
>
> To preserve my brains I want food and this is now my first consideration. Any sympathetic letter from you will be helpful to me here to get a scholarship either from the University or from Government. . . .

These passages from Ramanujan's letters will give a faint indication of the human side of a correspondence which stands as a remarkable one in the history

of our science. Along with the letters numerous astonishing mathematical results were communicated by Ramanujan to Hardy. Many of these were right; not a few were wrong; even the errors themselves were sometimes brilliant. The association thus opened up with Hardy became one of the prime factors in the life of Ramanujan. He went to England in 1914.

Hardy has put on record the fact that his appearance set a great puzzle for solution. What was to be done in teaching such an intellect the spirit and methods of modern mathematics? In some directions his knowledge was profound. In others his limitations were quite as startling. He could work out modular equations and theorems of complex multiplication, to orders unheard of. His mastery of continued fractions was remarkable. He had found for himself the functional equation of the Zeta-function and dominant terms in the case of many famous problems in the analytic theory of numbers. Yet he had never heard of a doubly periodic function or of Cauchy's theorem. Indeed, he had but the vaguest idea of a function of a complex variable. His conception of mathematical proof was quite inadequate. His results, new or old, right or wrong, had been obtained by argument, intuition, induction, mingled in a way which he himself could not coherently describe. Many of his proofs were invalid. Not a few of his results were false. His approximations were not as close as he supposed. But, notwithstanding the fact that he had never seen a French or German book and that his command of English was meager, he had conceived for himself and had treated in an astonishing way problems to which for a hundred years some of the finest intellects in Europe had given their attention without having reached a complete solution. That such an untrained mind made mistakes in dealing with such questions is not remarkable. What is astonishing is that it ever occurred to him to treat these problems at all.

In a few years after reaching England Ramanujan had a fair knowledge of the theory of functions and the analytic theory of numbers. He had learned to know when he had proved a theorem and when he had not. And his flow of original ideas kept up without abatement during this period of acquisition.

In the spring of 1917 he began to show evidence of being unwell. He went to a Nursing Home at Cambridge early in the summer of 1917 and was never afterwards out of bed for long at a time. He died April 26, 1920, a little more than seven years after writing his initial letter to Hardy, the event which brought him for the first time into touch with modern mathematical ideas. During this brief interval, about one-third of which was spent in bed on account of illness, he added to the achievements made unaided during the first years of his life a body of contributions which is sure to exert a marked influence on the development of certain chapters of mathematics, indeed has already exerted such an influence.

Under the title *Collected Papers of Srinivasa Ramanujan* there has been brought together in a single volume everything published by Ramanujan with the exception of a few solutions of questions proposed by other mathematicians and answered by him in the *Journal of the Indian Mathematical Society*. He left

behind him a large mass of unpublished material in note-books now famous. This is in process of being edited for publication.

Such is a brief statement concerning the human aspects of a remarkable mathematician some of whose contributions to the theory of numbers we shall now describe very briefly.

3. *On certain arithmetical functions.* We shall entirely pass over several of Ramanujan's contributions to the theory of numbers, including his investigation of the expression of integers in the form $ax^2+by^2+cz^2+dt^2$ and his theory of highly composite numbers (the latter work, in Hardy's judgment, belonging perhaps to a backwater of mathematics) and shall take up first a paper of 1916 relating to several arithmetical functions (*Collected Papers*, pp. 136–162).

Let $\sigma_s(n)$ denote the sum of the sth powers of the divisors of the positive integer n and let $\sigma_s(0)=\frac{1}{2}\zeta(-s)$, where $\zeta(s)$ is the Riemann Zeta-function $1^{-s}+2^{-s}+3^{-s}+\cdots$. Write

$$\sum_{r,s}(n) = \sigma_r(0)\sigma_s(n) + \sigma_r(1)\sigma_s(n-1) + \cdots + \sigma_r(n)\sigma_s(0).$$

Then Ramanujan proves that

$$\sum_{r,s}(n) = \frac{\Gamma(r+1)\Gamma(s+1)}{\Gamma(r+s+2)}\,\frac{\zeta(r+1)\zeta(s+1)}{\zeta(r+s+2)}\,\sigma_{r+s+1}(n) + \frac{\zeta(1-r)+\zeta(1-s)}{r+s}\,n\sigma_{r+s-1}(n) + O\{n^{2(r+s+1)/3}\},$$

whenever r and s are positive odd integers; and he shows that the error term denoted by O is zero in the following nine cases: $r=1$, $s=1, 3, 5, 7, 11$; $r=3$, $s=3, 5, 9$; $r=5$, $s=7$.

In the proof of this deep-lying result skillful use is made of transformations involving identities from the theory of trigonometric and elliptic functions. Many striking relations are obtained. In the course of the argument the author is led to introduce (among others) the number-theoretic function $\tau(n)$ defined by the identity

$$\sum_{n=1}^{\infty}\tau(n)x^n = x\{(1-x)(1-x^2)(1-x^3)\cdots\}^{24}.$$

He establishes several remarkable properties of this function and conjectures others. Later writers have pushed the investigation further.

It is shown that a similar analysis may be applied to interesting arithmetical functions of a different kind and in particular to the function $r_{2s}(n)$ which denotes the number of ways of representing the positive integer n as a sum of $2s$ squares. But it is convenient to treat this in connection with the analysis of the next paper.

4. *Certain trigonometric sums and expansions in infinite series.* The memoir of 1918 reprinted on pages 179–199 of Ramanujan's *Collected Papers* should be

read in connection with the one treated in §3. The two taken together contain a contribution of great interest. The later one has the merit that much of it is easy to read.

Let $c_s(n)$ denote the sum of the nth powers of the primitive sth roots of unity, so that

$$c_s(n) = \sum_{\lambda} \cos \frac{2\pi\lambda n}{s}$$

where λ is prime to s and not greater than s. Ramanujan's principal object in this paper is to obtain expressions for a variety of well-known arithmetical functions of n in the form of the series

$$\sum_s a_s c_s(n)$$

where a_s is independent of n. A typical formula is

$$\sigma(n) = \frac{\pi^2 n}{6}\left\{\frac{c_1(n)}{1^2} + \frac{c_2(n)}{2^2} + \frac{c_3(n)}{3^2} + \cdots\right\}$$

where $\sigma(n)$ is the sum of the divisors of n. Many of the formulas obtained are elementary in the technical sense that they can be proved by a combination of processes involving only finite algebra and simple general theorems concerning infinite series. But some are of a deeper character and can be proved only by means of theorems which seem to depend essentially on the theory of analytic functions. The remarkable formula

$$c_1(n) + \tfrac{1}{2}c_2(n) + \tfrac{1}{3}c_3(n) + \cdots = 0$$

is typical of the latter class.

It is convenient to employ the symbol $\eta_s(n)$ to denote the sum of the nth powers of all the sth roots of unity, whence $\eta_s(n) = s$ or 0 according as n is or is not a multiple of s. Then we have

$$\eta_s(n) = \sum_{\delta} c_\delta(n)$$

where δ runs over the divisors of s. Thence it follows that

$$c_s(n) = \sum_{\delta} \mu(\delta')\eta_\delta(n)$$

where δ is a divisor of s, $\delta\delta' = s$ and $\mu(\nu)$ is defined by the identity

$$\sum \mu(\nu)\nu^{-s} = \{\zeta(s)\}^{-1}.$$

Ramanujan also employs the function $s_q(n)$,

$$s_q(n) = \sum_{\lambda} (-1)^{(\lambda-1)/2} \sin \frac{2\pi\lambda n}{q}, \quad q = 1, 2, 3, \cdots,$$

where the sum is taken for the $\phi(q)$ integers which are prime to q and do not exceed q. He also uses the function

$$\gamma_q(n) = c_q(n) \cos \tfrac{1}{2}\pi s(q-1) - s_q(n) \sin \tfrac{1}{2}\pi s(q-1), \quad q = 1, 2, 3, \cdots,$$

where s is a given positive integer.

Using $\sigma_s(n)$ to denote the sum of the sth powers of the divisors of the positive integer n, he shows that

$$\sigma_s(n) = n^s \zeta(s+1)\left\{\frac{c_1(n)}{1^{s+1}} + \frac{c_2(n)}{2^{s+1}} + \frac{c_3(n)}{3^{s+1}} + \cdots\right\}, \quad s > 0.$$

Let $p_1, p_2, p_3, \cdots$, be the prime divisors of n and write

$$\phi_s(n) = n^s(1 - p_1^{-s})(1 - p_2^{-s})(1 - p_3^{-s}) \cdots.$$

Then $\phi_1(n) = \phi(n)$, the number of integers not greater than n and prime to n. Then Ramanujan proves that

$$\frac{\phi_s(n)\zeta(s+1)}{n^s} = \frac{\mu(1)c_1(n)}{\phi_{s+1}(1)} + \frac{\mu(2)c_2(n)}{\phi_{s+1}(2)} + \frac{\mu(3)c_3(n)}{\phi_{s+1}(3)} + \cdots.$$

If $r_{2s}(n)$ denotes the number of ways of representing the positive integer n as a sum of $2s$ squares and $\delta_{2s}(n)$ denotes the function

$$\delta_{2s}(n) = \frac{\pi^s n^{s-1}}{(s-1)!} g_s(n)$$

where

$$g_s(n) = \frac{\gamma_1(n)}{1^s} + \frac{\gamma_4(n)}{2^s} + \frac{\gamma_3(n)}{3^s} + \frac{\gamma_8(n)}{4^s} + \frac{\gamma_5(n)}{5^s} + \frac{\gamma_{12}(n)}{6^s} + \frac{\gamma_7(n)}{7^s} + \cdots, \quad s > 1,$$

and $g_1(n)$ denotes half of what the second member of the last equation becomes for $s = 1$, then, by aid of results in the paper described in our §3, Ramanujan shows that $r_{2s}(n) = \delta_{2s}(n)$ for $s = 1, 2, 3, 4$ and in general that

$$r_{2s}(n) = \delta_{2s}(n) + O(n^{s/2}).$$

This paper contains many other remarkable expansions of number-theoretic functions obtained by aid of the trigonometric sums $c_q(n)$, $s_q(n)$, $\gamma_q(n)$.

5. *Partitions of integers.* Let $p(n)$ denote the number of unrestricted partitions of n, that is, the number of ways of representing n as a sum of positive integers. Then, in accordance with a classic result, $p(n)$ is the coefficient of x^n in the expansion

$$f(x) = 1 + \sum_{n=1}^{\infty} p(n)x^n = \{(1-x)(1-x^2)(1-x^3)\cdots\}^{-1}.$$

Hardy has expressed the opinion that Ramanujan shows at his very best in the theory of partitions and the allied parts of the theories of elliptic functions and continued fractions. Besides his contributions in collaboration with Hardy (to be treated in §IV) and those in the papers already mentioned, the most important published work of Ramanujan in the theory of partitions is contained in the three articles appearing on pages 210–213, 214–215, 232–238 of his *Collected Papers*.

In the first of these papers Ramanujan conjectures the following theorem: If $\delta = 5^a 7^b 11^c$ and $24\lambda \equiv 1 \ mod\ \delta$, then

$$p(\lambda),\ p(\lambda+\delta),\ p(\lambda+2\delta),\ \cdots \equiv 0 \bmod \delta.$$

He found the theorem supported by all the available evidence but he was unable to obtain a general proof. He proved simply, however, that $p(5m+4) \equiv 0 \ mod\ 5$, $p(7m+5) \equiv 0 \ mod\ 7$. With greater difficulty he was able to prove some other independent cases of the conjectured theorem. In the course of this paper several identities were established, the most elegant of which is the following

$$p(4) + p(9)x + p(14)x^2 + \cdots = 5\frac{\{(1-x^5)(1-x^{10})(1-x^{15})\cdots\}^5}{\{(1-x)(1-x^2)(1-x^3)\cdots\}^6},$$

from which it appears directly that $p(5m+4)$ is divisible by 5. In speaking of the beautiful formulas in Ramanujan's work, Hardy says that if he had to select one from all of them he would agree with Major MacMahon in choosing this one.

The third paper just mentioned has intimate contacts with several other papers of Ramanujan. It contains new proofs of the two congruences just mentioned and the first published proof of the congruence $p(11m+6) \equiv 0 \ mod\ 11$.

6. *Remarks.* This very inadequate sketch of Ramanujan's contributions to the theory of numbers must fail to call forth any proper appreciation of the character of his labors. This can be gotten only from a first-hand contact with the work itself and from a realization of the stimulus which it imparts. We can not here take the space to follow up the extensions of Ramanujan's work except as they may appear incidentally in later sections.

II. REPRESENTATIONS AS SUMS OF SQUARES

7. *Orthogonal arithmetical functions.* A study of Ramanujan's trigonometric sums and of his expansions in terms of them has led me (in an unpublished paper) to introduce the general notion of orthogonal arithmetical functions. If two functions $f(n)$ and $g(n)$ are defined for $n = 1, 2, 3, \cdots$, they will be called orthogonal when (1) they have positive integral periods α and β and when (2) the sum $\sum f(n)g(n)$ (for $n = 1, 2, \cdots, l$) has the value 0, l being the least common multiple of α and β. If two (finite or infinite) sets of functions $f_1(n), f_2(n), \cdots$, and $g_1(n), g_2(n), \cdots$, are such that $f_i(n)$ and $g_j(n)$ are orthogonal when $i \neq j$, then the two sets are said to be biorthogonal. If the single set $f_1(n), f_2(n), \cdots$,

is such that every function in it is orthogonal to every other function in it, then it is said to be a set of orthogonal arithmetical functions.

For the named biorthogonal sets we shall be interested mainly in the case when the supplementary condition

$$\sum_{n=1}^{l_i} f_i(n)g_i(n) \neq 0$$

is satisfied for each i, where l_i is the least common multiple of the periods of $f_i(n)$ and $g_i(n)$. For the case of a single set the supplementary condition takes the form

$$\sum_{n=1}^{\alpha_i} f_i^2(n) \neq 0$$

where α_i is the period of $f_i(n)$. It is surely satisfied when $f_i(n)$, for every fixed i, is real-valued for all n and is not identically zero.

In the cases of both biorthogonal and orthogonal sets satisfying the supplementary condition there is an expansion theory, analogous to that of Fourier series, for the expansion of suitable arithmetical functions $h(n)$ in the form

$$h(n) = a_1f_1(n) + a_2f_2(n) + a_3f_3(n) + \cdots,$$

where the coefficients a_i are independent of n.

Ramanujan's trigonometric sums have the orthogonality properties expressed by the following relations:

$$\sum_{n=1}^{q\rho} c_q(n)c_\rho(n) = 0 \text{ if } \rho \neq q, \quad \sum_{n=1}^{q} c_q^2(n) = q\phi(q);$$

$$\sum_{n=1}^{q\rho} c_q(n)s_\rho(n) = 0 \text{ if } \rho \neq q;$$

$$\sum_{n=1}^{q\rho} s_q(n)s_\rho(n) = 0 \text{ if } \rho \neq q, \quad \sum_{n=1}^{q} s_q^2(n) = \tfrac{1}{4}q \sum_{\lambda} \left\{(-1)^{(\lambda-1)/2} - (-1)^{(q-\lambda-1)/2}\right\}^2$$

where λ runs over the integers prime to q and not exceeding q. In particular,

$$\sum_{n=1}^{4q} s_{4q}^2(n) = 4q\phi(4q).$$

The corresponding orthogonality properties of $\gamma_q(n)$ are implied by the foregoing.

8. *Sums of squares.* In a paper in the *Annals of Mathematics*, 1931, I have employed these orthogonality properties and certain deep-lying results due to Ramanujan in establishing several asymptotic relations which will be reproduced in §§8–10.

With the notation of the latter part of §4, we have

$$\frac{1}{m}\sum_{n=1}^{m}\gamma_\rho(n)n^{1-s}\delta_{2s}(n) = \frac{\pi^s\epsilon_\rho(s)\phi(\rho)}{(s-1)!\rho^s}+O\left(\frac{1}{m}\right),\quad s>3,$$

where $\epsilon_\rho(s)$ is 1, 0 or 2^s according as ρ is odd, $\rho\equiv 2 \bmod 4$ or $\rho\equiv 0 \bmod 4$; also,

$$\frac{1}{m}\sum_{n=1}^{m}\gamma_\rho(n)n^{1-s}r_{2s}(n) = \frac{\pi^s\epsilon_\rho(s)\phi(\rho)}{(s-1)!\rho^s}+C\left(\frac{1}{m}\right),\quad s>3.$$

The weight factors $\gamma_\rho(n)$ are suitable for smoothing out (in the limit) the irregularities of each of the functions $n^{1-s}r_{2s}(n)$ and $n^{1-s}\delta_{2s}(n)$. In particular, if we take $\rho=1$ we have the following general theorem: The functions $n^{1-s}r_{2s}(n)$ and $n^{1-s}\delta_{2s}(n)$ are each in the mean (on the average) equal to $\pi^s/(s-1)!$ if $s>3$, the error term for the average of the values $n=1, 2, \cdots, m$ being $O(1/m)$.

9. *Sums of triangular numbers.* If $R_{2s}(n)$ denotes the number of ways of representing n as a sum of $2s$ triangular numbers, where s is a given positive integer, and if we write $R_{2s}(n)=D_{2s}(n)+E_{2s}(n)$, where

$$D_{2s}(n) = \frac{(\frac{1}{2}\pi)^s}{(s-1)!}(n+\tfrac{1}{4}s)^{s-1}\sum_{\mu=1}^{\infty}(2\mu-1)^{-s}(-1)^{(\mu-1)s}c_{2\mu-1}\left(\frac{4n+s}{\eta(s)}\right),\quad s>1,$$

$\eta(s)$ being 1, 2 or 4 according as s is odd, is twice an odd number or is a multiple of 4, while $D_2(n)$ is half of what the last second member becomes for $s=1$, then Ramanujan has shown that $E_{2s}(n)=0$ for $s=1, 2, 3, 4$ while in general $R_{2s}(n)\sim D_{2s}(n)$. We have the following mean value relations:

$$\frac{1}{m}\sum_{n=1}^{m}c_{2\lambda-1}\left(\frac{4n+s}{\eta(s)}\right)(n+\tfrac{1}{4}s)^{1-s}D_{2s}(n) = \frac{(-1)^{(\lambda-1)s}(\frac{1}{2}\pi)^s\phi(2\lambda-1)}{(s-1)!(2\lambda-1)^s}+O\left(\frac{1}{m}\right),\quad s>3,$$

$$\frac{1}{m}\sum_{n=1}^{m}c_{2\lambda}\left(\frac{4n+s}{\eta(s)}\right)(n+\tfrac{1}{4}s)^{1-s}D_{2s}(n) = O\left(\frac{1}{m}\right),\quad s>3,\ s\equiv 0 \bmod 4,$$

where in each case λ ranges over the set $1, 2, 3\cdots$.

10. *Ramanujan's function* $\sum_{r,s}(n)$. This function (defined in §3) has the following property:

$$\frac{1}{m}\sum_{n=1}^{m}c_\rho(n)n^{-r-s-1}\sum_{r,s}(n) = \frac{\Gamma(r+1)\Gamma(s+1)}{\Gamma(r+s+2)}\,\frac{\phi(\rho)\zeta(r+1)\zeta(s+1)}{\rho^{r+s+2}}+O\left(\frac{\log m}{m}\right)$$

whenever r and s are positive odd integers.

III. EXPANSIONS OF ARITHMETICAL FUNCTIONS IN INFINITE SERIES

11. *Generalizations of Ramanujan's trigonometric sums.* By a character $\chi(a)$ mod k we mean (as usual) a function $\chi(a)$ such that (1) $\chi(a)=0$ if a is not prime to k, (2) $\chi(1)=1$, (3) $\chi(a_1a_2)=\chi(a_1)\chi(a_2)$, (4) $\chi(a_1)=\chi(a_2)$ if $a_1\equiv a_2 \bmod k$. By $\chi(a)$, $\chi_1(a)$, $\chi_2(a)$, $\cdots$, we denote any characters moduli k, k_1, k_2, $\cdots$, respectively, and by $\chi_0(a)$, $\chi_{10}(a)$, $\chi_{20}(a)$, $\cdots$, we denote the principal characters

for these moduli, the principal character $\chi(a)$ modulo k being that character for which $\chi(a)=1$ or 0 according as a is or is not prime to k. If a symbol for a character is written with an argument which is not an integer it may have for this argument any conveniently assigned value, since it will appear in our formulas only when multiplied by a quantity having the value zero.The character which is equal to unity for all arguments will sometimes be replaced by 1. We use $\mu(a)$ and $\eta_q(n)$ in the senses defined in §4.

We introduce the functions

$$c_q^{(\lambda)}(n, \chi, \chi_1, \chi_2) = \sum_{d|q} \mu\left(\frac{q}{d}\right)\chi\left(\frac{q}{d}\right)\chi_1(d)\chi_2\left(\frac{n}{d}\right)\eta_d^{\lambda}(n)$$

where the sum is taken for the divisors d of q and where λ is positive. These functions reduce to Ramanujan's function $c_q(n)$ when $\lambda=1$ and $\chi(a)\equiv\chi_1(a)\equiv\chi_2(a)\equiv 1$. These new functions are rich in properties generalizing many of the properties of Ramanujan's functions $c_q(n)$. In particular, important expansions of number-theoretic functions may be derived by their aid, as I have shown in *Proc. Nat. Acad. Sc.* for September 1930 and in an unpublished paper. A few of the results obtained by aid of these expansions are given in the next two sections.

Incidental to these investigations the following remarkable relation was discovered:

$$\sum_{q=1}^{\infty} \frac{1}{q} c_q^{(\lambda)}(n, \chi_0, \chi_1, \chi_2) = 0.$$

When $\lambda=1$ and $\chi_0(a)\equiv\chi_1(a)\equiv\chi_2(a)\equiv 1$ the foregoing equation expresses a deep-lying result due to Ramanujan (see our §4).

12. *Functions of the divisors of an integer.* We introduce the function

$$\sigma_s(n, \chi_1, \chi_2) = \sum_{d|n} \chi_1(d)\chi_2\left(\frac{n}{d}\right)d^s$$

where the sum is taken for the divisors d of n. Here s denotes the complex variable $\sigma+it$ where σ and t are real. By $L(s, \chi)$ we denote the L-series

$$L(s, \chi) = \sum_{a=1}^{\infty} \frac{\chi(a)}{a^s}.$$

Then if $\sigma+\lambda>1$ we have the infinite expansions

$$\sigma_s(n, \chi_1, \chi_2) = n^s L(s+\lambda, \chi) \sum_{\nu=1}^{\infty} \frac{c_\nu^{(\lambda)}(n, \chi, \chi_2, \chi_1)}{\nu^{s+\lambda}},$$

$$\sigma_{-s}(n, \chi_1, \chi_2) = L(s+\lambda, \chi) \sum_{\nu=1}^{\infty} \frac{c_\nu^{(\lambda)}(n, \chi, \chi_1, \chi_2)}{\nu^{s+\lambda}}.$$

Numerous consequences may readily be drawn from these formulas. In particular, we have

$$\frac{1}{m}\sum_{n=1}^{m} c_\rho(n)\sigma_{-s}(n, 1, 1) = \frac{\phi(\rho)}{\rho^{s+1}}\left(1 + \frac{1}{2^{s+1}} + \frac{1}{3^{s+1}} + \cdots\right) + O\left(\frac{1}{m}\right), \sigma > 2,$$

$$= \frac{\phi(\rho)}{\rho^{2\lambda}}\,\pi^{2\lambda}\,\frac{2^{2\lambda-1}B_\lambda}{(2\lambda)!} + O\left(\frac{1}{m}\right) \text{ if } s = 2\lambda - 1,$$

where in the last member λ is an integer greater than unity and the B's are the Bernoulli numbers $B_1 = 1/6$, $B_2 = 1/30$, $\cdots$. More generally, we have

$$\frac{1}{m}\sum_{n=1}^{m} c_\rho(n)\sigma_{-s}(n, \chi, 1) = L(s + 1, \chi)\rho^{-s-1}\chi(\rho)\phi(\rho) + O(1/m), \sigma > 2,$$

an asymptotic relation special cases of which have appeared in several papers (see Dickson's *History of the Theory of Numbers*, Vol. I, pp. 291–4, 301, 322).

We have $\sigma_{-s}(n, \chi_0, 1) = \gamma_{-s}(n, k) =$ the sum of the sth powers of the reciprocals of those divisors of n which are prime to the modulus k. It may be shown that

$$\frac{1}{m}\sum_{n=1}^{m} c_\rho(n)\gamma_{-s}(n, k) = \frac{\chi_0(\rho)\phi(\rho)}{\rho^{s+1}}\sum_{q}{}' \frac{1}{q^{s+1}} + O\left(\frac{1}{m}\right), \sigma > 2,$$

where the prime in $\sum'$ indicates that q ranges over the integers which are prime to k.

To illustrate the meaning of these formulas we take $\rho = 1$ in the third preceding equation and obtain the following theorem:

When λ is an integer greater than unity, the sum of the $(2\lambda-1)$th powers of the reciprocals of the divisors of n is in the mean (on the average) equal to $\pi^{2\lambda}2^{2\lambda-1}B_\lambda/(2\lambda)!$, the error term of the average for $n = 1, 2, \cdots, m$ being $O(1/m)$.

13. *Representations as sums of squares.* The foregoing properties of σ-functions are intimately connected with expansions of other number-theoretic functions. We merely state three particular results. Using $N[n = x^2 + y^2]$ to denote the number of ways of representing n as a sum of two squares and employing similar notations for other cases, we have

$$N[n = x^2 + y^2] = 2\pi\left\{c_1(n) + \frac{1}{5}c_5(n) + \frac{1}{9}c_9(n) + \cdots\right\},$$

$$N[n = x^2 + 2y^2] = \pi(2)^{1/2}\left\{c_1(n) + \frac{1}{3}c_3(n) + \frac{1}{9}c_9(n) + \frac{1}{11}c_{11}(n) + \cdots\right\},$$

$$N[2^\alpha m = x^2 + y^2 + z^2 + t^2, m \text{ odd}] = g(\alpha)\pi^2 m\left\{c_1(m) + \frac{c_3(m)}{3^2} + \frac{c_5(m)}{5^2} + \cdots\right\},$$

where $g(\alpha) = 1$ or 3 according as $\alpha = 0$ or $\alpha > 0$ and where in the second formula the subscripts appearing are those of the forms $8x+1$ and $8x+3$.

IV. THEORY OF PARTITIONS

14. *Older and newer problems.* From the time of Euler onwards to 1917 and 1918 when Hardy and Ramanujan (Ramanujan's *Collected Papers*, pp. 239–241, 244, 276–309) took up the problem, the theory of the partition of integers into positive integral summands had been developed almost entirely from an algebraic point of view. The results consisted mainly of formal identities, many of which are quite ingenious and of beautiful character. Of genuine asymptotic formulas it is fair to say that there were none, the only exceptions to this statement consisting of special results mainly of algebraic interest and essentially trivial from the point of view of asymptotic formulas. So true indeed is this statement that Hardy and Ramanujan found in the literature of the subject no allusion whatever to the question of the order of magnitude of the fundamental partition function $p(n)$, where $p(n)$ denotes the number of unrestricted partitions of n and is the coefficient of x^n in the expansion of the function

$$f(x) = 1 + \sum_{n=1}^{\infty} p(n)x^n = \{(1 - x)(1 - x^2)(1 - x^3) \cdots\}^{-1}.$$

From Cauchy's theorem we have

$$p(n) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(x)}{x^{n+1}}\, dx$$

where Γ is a path which lies within the unit circle and encloses the point 0. The dominant idea in the method of Hardy and Ramanujan lies in the use of Cauchy's integral formula to obtain asymptotic properties of $p(n)$. This idea, extremely obvious as it is and dominating as it does a very large part of the modern analytic theory of numbers, appears never before to have been employed in the problem of partitions. Perhaps one reason for this is in the fact that since the time of Euler those who have investigated the theory of partitions have been interested primarily in algebra. But a more important reason lies in the extreme complexity of the generating function $f(x)$ near a point on the unit circle. The function does not exist outside of this circle, and every point of the circle is an essential singularity of the function. No part of the contour of integration can be so deformed as to become obviously negligible; every element of it requires special study; there is no obvious method of obtaining a dominant term in the value of $p(n)$.

This makes the difficulties appear very serious. But there is another fact which could lead one to hope for success. The formulas of the theory of the linear transformation of the elliptic functions afford a very powerful tool for studying the behavior of $f(x)$ near an assigned point of the unit circle. It is to an appropriate use of these formulas that the authors owe the startling success of their investigation.

15. *Elementary results.* Hardy and Ramanujan treat the problem of partitions not only by means of the deep-lying methods just referred to but also by

others of a more elementary character. In such problems it is desirable to distinguish clearly the stages to which progress can be made by progressively more deep-lying methods. The earlier results so obtained are likely to be completely superseded by the later; but it may be expected that the more elementary methods will often admit applications to other problems in which the more subtle analysis is impracticable.

By means of quite elementary methods depending on the Euler identity

$$\frac{1}{(1-x)(1-x^2)(1-x^3)\cdots} = 1 + \frac{x}{(1-x)^2} + \frac{x^4}{(1-x)^2(1-x^2)^2} + \cdots .$$

Hardy and Ramanujan first prove the existence of positive constants A and B such that for all sufficiently large values of n we have

$$An^{1/2} < \log p(n) < Bn^{1/2}.$$

This raises the question whether there exists a constant C such that $\log p(n) \sim Cn^{1/2}$. By methods more difficult but not appealing to the theory of analytic functions they show that the answer is affirmative and that the value of C is $C=\pi(2/3)^{1/2}$. This means that

$$p(n) = \exp\left\{\pi(2n/3)^{1/2}(1+\epsilon)\right\}$$

where ϵ is small when n is large.

The next step of Hardy and Ramanujan was to replace the last result by the more precise formula

$$p(n) \sim (4n\cdot 3^{1/2})^{-1} \exp\left\{\pi(2n/3)^{1/2}\right\}.$$

For the proof of this result they required more powerful machinery, including the Cauchy integral formula for $p(n)$ already given and an important functional relation arising in the theory of elliptic functions. (This formula was independently discovered by Uspensky in 1920.)

16. *Asymptotic character of* $p(n)$. To obtain still more satisfactory results they found it necessary to construct an auxiliary function $F(x)$ regular on the unit circle except at $x=1$ and having there a singularity as nearly as possible like that of $f(x)$ at $x=1$. This procedure was suggested by the fact that though every point of the unit circle is a singularity of $f(x)$ yet in a certain sense the point $x=1$ is much the heaviest singularity. They hoped then to attain the goal by applying Cauchy's theorem to $f(x)-F(x)$ instead of to $f(x)$. Their hope was justified and they obtained the result

$$p(n) \sim (2\pi\cdot 2^{1/2})^{-1}\frac{d}{dn}(e^{C\lambda_n}/\lambda_n) + O(e^{Dn^{1/2}})$$

where $C=\pi(2/3)^{1/2}$, $\lambda_n=(n-1/24)^{1/2}$ and D is any number greater than $\frac{1}{2}C$.

Since the error term O in this formula is of exponential type and might therefore be expected ultimately to increase with very great rapidity, the authors were surprised to find that the formula gave exceedingly good results for fairly large values of n. They were therefore encouraged to push the approximation further. As they did so and as the successive formulas met in a striking way the test of numerical data, the authors were led to look for much more than they had first expected and indeed to seek a formula for $p(n)$ which not only exhibits its order of magnitude but may be used to compute its exact value. They succeeded in obtaining precisely this startling result.

During the process of their investigation the authors had the use of numerical data supplied by Major MacMahon and Mr. H. B. C. Darling; they acknowledge particular indebtedness to the former. They add: "It is certain that, without the encouragement given by the results of these calculations, we should never have attempted to prove theoretical results at all comparable in precision with those which we have enunciated."

The main theorem obtained is the following:

Suppose that

$$\phi_q(n) = \frac{1}{2\pi}(\tfrac{1}{2}q)^{1/2}\frac{d}{dn}(e^{C\lambda_n/q}/\lambda_n),\ C = \pi(\tfrac{2}{3})^{1/2},\ \lambda_n = (n - 1/24)^{1/2}$$

for all positive integers q; that p is a positive integer less than q and prime to q; that $\omega_{p,q}$ is a $(24q)$th root of unity, defined when p is odd by the formula

$$\omega_{p,q} = \left(\frac{-q}{p}\right)\exp\left[-\left\{\frac{1}{4}(2 - pq - p) + \frac{1}{12}\left(q - \frac{1}{q}\right)(2p - p' + p^2p')\right\}\pi i\right],$$

and when q is odd by the formula

$$\omega_{p,q} = \left(\frac{-p}{q}\right)\exp\left[-\left\{\frac{1}{4}(q - 1) + \frac{1}{12}\left(q - \frac{1}{q}\right)(2p - p' + p^2p')\right\}\pi i\right],$$

where (a/b) is the symbol of Legendre and Jacobi, and p' is any positive integer such that $1+pp'$ is divisible by q; that

$$A_q(n) = \sum_p \omega_{p,q}e^{-2np\pi i/q};$$

and that α is any positive constant and ν is the integral part of $\alpha n^{1/2}$.

Then

$$p(n) = \sum_{q=1}^{\nu} A_q(n)\phi_q(n) + O(n^{-1/4}),$$

so that $p(n)$ is, for all sufficiently large values of n, the integer nearest to

$$\sum_{q=1}^{\nu} A_q(n)\phi_q(n).$$

V. WARING'S PROBLEM

17. *Formulation and earlier history of the problem.* Let us consider the problem of the representation of positive integers n as sums of positive integral kth powers. There is always a solution since n is a sum of n ones. For the moment let us consider k fixed. When n is given there is then a *minimum* number of kth powers of which n is the sum. Thus 23 is the sum of nine cubes ($23=2.2^3+7.1^3$) but of no smaller number. When k is fixed and n varies over all positive integers in order of increasing magnitude, will the minimum number of kth powers with sum n remain bounded? It does so when k is 2; for then it never exceeds 4. Waring's problem, in the restricted sense, is the problem of proving that this minimum number remains bounded in the general case.

To state the problem formally, let k be given. If a number m exists such that every positive integer n is a sum of not more than m kth powers, then there is a *least* value of m for which this is true. This least number, if it exists, depends on k and is uniquely determined when k is given. We denote it by $g(k)$. Thus, by definition, $g(k)$ is the least number, if such a number exists, for which it is true that every positive integer is the sum of not more than $g(k)$ kth powers. We have seen that $g(2)=4$.

In his *Meditationes Algebraicae* (Cambridge, 1770, pp. 204–5; 3rd edition, 1782, pp. 349–350) Waring stated that every positive integer is a sum of at most four integral squares, of at most 9 positive integral cubes, of at most 19 integral fourth powers, and so forth. There is a lack of precision concerning what is here meant by the words of continuation, but it appears that Waring intended to assert precisely the existence of $g(k)$. He implies that $g(2)=4$, $g(3)=9$, $g(4)=19$. The first had already been proved by Lagrange. The others we shall discuss later.

Let $[\alpha]$ denote the integral part of α. It was observed early and is easily verified that the number $([(3/2)^k]-1)2^k+2^k-1$, which is less than 3^k, requires at least $[(3/2)^k]+2^k-2$ kth powers for its representation, 2^k-1 of them being ones and the remaining being each 2^k. For $k=2,3,\cdots$, we therefore have 4, 9, 19, 37, 73, 143, 279, $\cdots$, as lower bounds for $g(2)$, $g(3)$, $\cdots$.

Waring gave no argument to support his statement. It seems that he intended to offer it only as a highly plausible speculation. But what is needed is proof; and that it has been terribly difficult to supply. In the time of Waring it was known that $g(2)=4$. In 1859 Liouville proved the existence of $g(4)$ and that its value is not greater than 53; his method is quite simple. The existence of $g(3)$ was first established in 1895 by Maillet who proved that $g(3)\leqq 17$. Other particular results followed by special methods. Waring's restricted problem of proving the existence of $g(k)$ for every positive integer k was first completely solved by Hilbert in 1909, 139 years after its formulation by Waring.

The numbers 23 and 239 require nine cubes each for their representation. There is no other number known requiring so many, and the matter has been tested numerically up to at least 40,000. In 1909 Landau proved the very beauti-

ful theorem that the number of positive integers requiring as many as 9 cubes is finite. This suggests the introduction of the number $G(k)$ defined as being the least number for which it is true that every number from a certain point onwards is a sum of not more than $G(k)$ kth powers. Since $g(k)$ exists it follows that $G(k)$ exists and that $G(k) \leqq g(k)$. The number $G(k)$ appears to be more fundamental than $g(k)$.

It may be further asked: What is the least integer $G_1(k)$ such that nearly all positive integers n are each expressible as a sum of not more than $G_1(k)$ positive integral kth powers? [To say that "nearly all" positive integers have a given property is to say that $\psi(n)/n \to 1$ as n becomes infinite where $\psi(n)$ is the number of positive integers not exceeding n and having the property in consideration.] From what precedes it is clear that $G_1(k)$ exists and that $g(k) \geqq G(k) \geqq G_1(k)$. It may be shown that $G_1(2) = 4$.

Hardy and Littlewood (see *Mathematische Zeitschrift* 23 (1925): 1–37 and the papers there cited) in recent years have made very remarkable contributions to the study of Waring's problem in the larger sense, namely, the investigation of the magnitude of the numbers $g(k)$, $G(k)$, $G_1(k)$. In §§18 and 19 we shall very briefly describe their work.

18. *The methods of Hardy and Littlewood.* In order to come to close grips with Waring's problem in the larger sense, to get into real contact with the actual values of $g(k)$, $G(k)$, $G_1(k)$, still more in order to treat the problem of the number of representations of a given integer as a sum of s kth powers, it is necessary to have powerful tools and to have in reach the means of handling them effectively. Only the modern theory of functions can be expected to yield the necessary assistance. It was from such considerations as these that Hardy and Littlewood were led to seek means of applying Cauchy's integral theorem to the problem; and it is through such application that they have obtained their results.

Let $r(n, k, s)$ denote the number of representations of n as a sum of s kth powers. Then, with suitable conventions concerning the way in which the number of representations is to be counted, we have

$$1 + \sum_{n=1}^{\infty} r(n, k, s)x^n = \{f(x)\}^s \equiv (1 + 2x^{1^k} + 2x^{2^k} + 2x^{3^k} + \cdots)^s.$$

The two infinite series occurring here are convergent when $|x| < 1$. Hence we have

$$r(n, k, s) = \frac{1}{2\pi i} \int_{\Gamma} \frac{\{f(x)\}^s}{x^{n+1}} dx$$

where Γ is a closed contour lying entirely within the unit circle and enclosing the point 0.

The study of this integral is intricate and difficult. Every point of the unit circle is a singularity of $f(x)$ of a very complicated kind and the circle itself is a

barrier beyond which the function does not exist. In his Oxford Inaugural Lecture Hardy has described as follows the fundamental idea employed in overcoming the difficulty:

> Among the continuous mass of singularities which covers up the circle, it is possible to pick out a class which to a certain extent dominates the rest. These special singularities are those associated with the *rational* points of the circle, that is to say, the points
>
> $$x = e^{2p\pi i/q},$$
>
> where p/q is a rational fraction in its lowest terms. . . . In the neighbourhood of these points the behaviour of the function is, sufficiently complex indeed, but simpler than elsewhere. The function has, to put the matter in a rough and popular way, a general tendency to become large in the neighbourhood of the unit circle, but this tendency is most pronounced near these particular points. They are not only the *simplest* but also the *heaviest* singularities; their weight is greatest when the denominator q is smallest, decreases as q increases, and (as a physicist would say) becomes infinitely small when q is infinitely large. There is, therefore, at any rate, the hope that we may be able to isolate the contributions of each of these selected points, and obtain, by adding them together, a series which may give a genuine approximation to our coefficient.

While this shows the point of departure of Hardy and Littlewood it can give no indication of the brilliance with which they have carried the program through to the results now to be stated.

19. *The results of Hardy and Littlewood.* With respect to the magnitude of $G_1(k)$ Hardy and Littlewood have proved the following theorem: If $k=3$ or $k \geqq 5$ then almost all positive integers are sums of

$$(\tfrac{1}{2}k - 1)2^{k-1} + 3$$

non-negative integral kth powers, and in particular of 5 cubes, 27 fifth powers, and 67 sixth powers; almost all positive integers are sums of 15 biquadrates, and $G_1(4) = 15$.

With respect to $G(k)$ they have the theorem: If $k > 3$ then all large positive integers are sums of

$$(\tfrac{1}{2}k - 1)2^{k-1} + k + 5 + \left[\frac{(k-2)\log 2 - \log k + \log (k-2)}{\log k - \log (k-1)}\right]$$

non-negative integral kth powers, where $[\alpha]$ denotes the integral part of α; and in particular of 19 biquadrates, 41 fifth powers, and 87 sixth powers.

The case of biquadrates is particularly interesting. Since $79 (= 4.2^4 + 15.1^4)$ actually requires 19 biquadrates, so that $g(4) \geqq 19$, the problem of determining the exact value of $g(4)$ is thus reduced (in theory at any rate) to a problem of computation.

From the work of Hardy and Littlewood, of Landau and of Wieferich it is known that

$$g(3) = 9, G(3) \leqq 8, \text{ and } G_1(3) \leqq 5.$$

It is known also that

$$\limsup_{k=\infty} g(k)/(k2^{k-1}) \leqq$$

It has also been proved that

$$G(k) \geqq G_1(k) \geqq k + 1 \text{ if } k \geqq 2.$$

A lower bound to $g(k)$ is given in §17.

Lack of space forbids our giving Hardy and Littlewood's results concerning the asymptotic character of the number $r(n, k, s)$ of ways of representing n as a sum of s kth powers.

VI. A QUESTION RELATED TO WARING'S PROBLEM[1]

20. *Multiplicative domains numerically defined.* Separate the prime numbers into k classes $C_1, C_2, \cdots, C_k$ and associate with the class C_i a positive integer α_i for $i = 1,2, \cdots, k$. Denote by D the multiplicative domain composed of 0, 1, the α_ith power of every prime in C_i $(i = 1, 2, \cdots, k)$ and all the integers which may be obtained from these generators by multiplication (with repetition of factors permissible). If μ is the least common multiple of $\alpha_1, \alpha_2, \cdots, \alpha_k$, then D contains the μth power of every positive integer. It also contains other integers unless $\alpha_1 = \alpha_2 = \cdots = \alpha_k$.

Since every positive integer is representable as a sum of $\nu(\mu)$ μth powers of non-negative integers, where $\nu(\mu)$ depends on μ alone, it follows that for every domain D there exists an integer $N(D)$, not greater than $\nu(\mu)$, such that every positive integer is representable as a sum of $N(D)$ integers belonging to D. The problem of determining $N(D)$, and other related integers suggested by the known facts concerning Waring's problem, is perhaps of little interest in the general case. But there is a special class of domains D for which the problem appears to be not devoid of interest.

21. *The domains D_m.* Let m be an integer greater than unity. Let D_m denote the domain D which has the following generators: 0, 1, every prime factor of m, and the τth power of every prime of the form $mx + \alpha$ where α ranges over the integers less than m and prime to m, and where τ $(= \tau_\alpha)$ is the exponent to which α belongs modulo m, so that $t = \tau$ is the least positive integer t such that $\alpha^t \equiv 1 \bmod m$. For such a domain D_m we propose the following questions (suggested by Waring's problem). What is the least integer $g(m)$ such that every positive integer is a sum of $g(m)$ integers belonging to D_m? What is the least integer $G(m)$ such that all but a finite number of positive integers are each expressible as a sum of $G(m)$ integers belonging to D_m? What is the least integer $G_1(m)$ such that nearly all positive integers are each expressible as a sum of $G_1(m)$ integers belonging to D_m? It is obvious that $g(m) \geqq G(m) \geqq G_1(m)$.

In the paper referred to I have established the following results concerning the values of these numbers:

I. $g(2) = G(2) = G_1(2) = 1$.
II. $g(3) = G(3) = G_1(3) = 2$.
III. $g(4) = G(4) = G_1(4) = 2$.

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[1] *Quarterly Journal of Mathematics* 2 (1931): 59–67.

IV. $g(6)=G(6)=G_1(6)=2$.
V. $g(8)=G(8)=G_1(8)=3$.
VI. $g(12)=3$, $G(12)=2$ or 3, $G_1(12)=2$ or 3.
VII. $g(24)=G(24)=G_1(24)=3$.

In no other cases have I succeeded in finding the values of $g(m)$, $G(m)$ and $G_1(m)$. These cases (except for the trivial case $m=2$) are characterized by the fact that they involve just those values of m which are such that 2 is the greatest exponent modulo m of any number prime to m, whence it follows that in these cases D_m contains every square. It is this fact which renders the treatment of these cases easy. I am unable to determine whether 2 or 3 is the correct value of either $G(12)$ or $G_1(12)$.

The remaining domains D_m which contain every fourth power are those for which $m=5$, 10, 15, 16, 20, 30, 40, 48, 60, 80, 120, 240. These seem to be the most promising ones for a further next investigation. It is easy to show that $G_1(p)\geqq p-1$ when p is any prime number, whence $G_1(5)\geqq 4$. There is some empirical evidence for believing that $g(5)=5$. More remarkable is the fact that every number up to 600 is a sum of two numbers in D_{10}; this raises the (as yet unanswered) question whether $g(10)=G(10)=G_1(10)=2$. Further empirical facts are given in the paper mentioned.

VII. GOLDBACH'S CONJECTURE

22. *The correspondence of Goldbach and Euler.* On June 7, 1742, Goldbach wrote to Euler that he considered it not without value that one should call attention to propositions which are very probable even though one is unable to give an actual demonstration and he hazarded the conjecture that every positive integer which is a sum of two primes is also a sum of as many primes as one wishes (unity being reckoned among the primes) up to a number of summands equal to the number itself. In his reply of June 30, 1742, Euler remarked that the foregoing proposition is easily proved if one grants a conjecture previously communicated to Euler by Goldbach, namely, that every even positive integer is a sum of two primes. Thus Euler definitely states that Goldbach had communicated to him the proposition (now known as Goldbach's conjecture) that every even number is a sum of two primes. But I do not know of any extant letter in which Goldbach actually states the theorem, though he did say (in his letter of June 7, 1742) that it appears that every number greater than two is an aggregate of three primes. Concerning the proposition that every even number is a sum of two primes, Euler remarked that he considered it a quite certain theorem although he was not able to demonstrate it.

On Nov. 18, 1752, Goldbach submitted to Euler the empirical proposition that every odd number is of the form $p+2a^2$ where p is a prime and a is a positive integer or zero. On Dec. 16, 1752, Euler replied that he had verified this theorem for all odd numbers up to 1000 and asks if the number of representations in the given form increases as the odd number itself increases. After some

further discussion, he adds: "Therefore I believe this theorem, although I would not swear to it." It is well that he would not swear to it if he is concerned not to perjure himself, since the theorem fails for $5777 = 53.109$.

23. *Early history of the Goldbach conjecture.* Descartes stated that every even number is a sum of 1, 2 or 3 primes. Waring says that every even number is a sum of two primes and every odd number is either a prime or the sum of three primes. Lagrange stated some related empirical propositions. A. de Polignac conjectured that every even number is the difference of two primes and in fact that it is a difference of two consecutive primes in an infinite number of ways. Many other empirical theorems have been stated which might be associated with these, but we shall not take time to give them.

Desboves (*Nouv. Ann. Math.* 14 (1855): 281–4) formulated the following conjecture (verified up to 10,000): Every even number except 2 is the sum of two prime numbers in at least two ways; and, when the even number is the double of an odd number, it is always simultaneously the sum of two primes of the form $4n+1$ and the sum of two primes of the form $4n-1$. The theorem as thus formulated has contacts with other propositions. It implies that between n and $2n$ ($n>6$) there are always at least two prime numbers. That every positive integer is a sum of four squares is implied by the theorem and the fact that a prime of the form $4n+1$ is a sum of two squares.

The problem has been considered also by Sylvester, Stäckel, Merlin and Brun in papers which set forth conjectured or otherwise insecure or incomplete results. For an analysis of them see the first paper by Hardy and Littlewood referred to in our §24.

In 1912 the state of Goldbach's conjecture and related matters was still such that Landau, in his address before the Fifth International Congress at Cambridge, pronounced inaccessible (unangreifbar) in the state of science at that time the questions: Is every even integer greater than 2 the sum of two primes? Is there an infinite number of prime-pairs whose difference is 2? In the remainder of this paper we shall see something of the extent to which such questions have since become accessible.

24. *The work of Hardy and Littlewood.* In two path-breaking memoirs (*Acta Math.* 44 (1922): 1–70, *Proc. Lond. Math. Soc.* (2) 22 (1923): 46–56) on the expression of a number as a sum of primes, Hardy and Littlewood attack the named problem with the aid of their new transcendental method (already partially described in our earlier sections), show that it can be brought into contact with the recognized methods of the analytic theory of numbers and say therefore that the problem is not "unangreifbar." They do not solve the problem; they do not even prove that every number is the sum of not more than a million or any other number of primes. In fact, as they themselves say, all the results of their memoirs (except certain incidental ones), so far as they are novel, depend upon the truth of a hypothesis which they assume but can not establish.

Have they thus rendered accessible such questions as those which Landau pronounced inaccessible in 1912? In his *Vorlesungen über Zahlentheorie*, 1927,

Landau (Vol. I, p. 183) replies that history has found him partly right and partly wrong, but that he finds himself quite right. What Hardy and Littlewood did in 1922 and 1923 leaves him unqualifiedly right in what he said about the state of science in 1912. But Landau holds that he was still right in 1927 since Hardy and Littlewood, though they have proved everything which they claimed to prove, have nevertheless done nothing more than to show the consequences (remarkable enough indeed) of a hypothesis which really consists of an infinite number of assumptions. (To guard against a misunderstanding of this friendly and good-natured controversy I need only remind you that Landau, in common with all who have read the Hardy-Littlewood memoirs, has a very high appreciation of their importance.)

In order to state the main results of Hardy and Littlewood it is necessary to employ the functions $L(s, \chi)$ defined by the series

$$L(s, \chi) = \sum_{a=1}^{\infty} \frac{\chi(a)}{a^s},$$

one such function being formed for each positive integer k and each character $\chi(a)$ modulo k. Here s is the complex variable $\sigma+it$ where σ and t are real. The unproved hypothesis relates to the distribution of the zeros of each of the infinitude of functions $L(s, \chi)$.

The main Hardy-Littlewood theorems (together with their hypothesis) may now be stated as follows:

If no root of any whatever $L(s, \chi)$ belongs to the half-plane $\sigma > \frac{1}{2}$ (and indeed if there exists an absolute constant C less than $3/4$ such that no root of any whatever $L(s, \chi)$ belongs to the half-plane $\sigma > C$), then

I. Every sufficiently large odd number is a sum of three odd primes;

II. Nearly every even positive number is a sum of two odd primes;

III. The number $N(n)$ of ways of representing the odd number n $(n>1)$ as a sum of three odd primes is of the form

$$N(n) = \frac{n^2}{2\log^3 n}\left\{ \prod_{p}\left(1+\frac{1}{(p-1)^3}\right)\cdot \prod_{p|n}\left(1-\frac{1}{p^2-3p+3}\right)+\lambda(n)\right\}$$

where $\lim_{n=\infty}\lambda(n)=0$ and the first product is taken for all odd primes p while the second is taken for the odd prime divisors of n.

The character of the investigation is indicated by these three fundamental theorems; but, taken alone, they do not give a clear indication of the richness of the results. There is a treatment of each of the following problems: the representation of integers as sums of r primes; the existence of an infinitude of prime pairs whose difference in each case is any preassigned even number k; generalizations of the last problem; the existence of an infinitude of primes of the form m^2+1; representation of a large non-square number as the sum of a prime and a square; representation of a large number as the sum of a prime and the double

of a square; and many others. In much of this work conjectures only are formulated but they are conjectures which come into close relation with the general methods of the memoir.

VIII. SCHNIRELMANN'S THEOREMS

25. *Character of Schnirelmann's methods.* Landau, who has given an exposition of Schnirelmann's theorems with some simplifications (Göttingen Nachrichten, 1930), says that the memoir from which the theorems are taken contains one of the greatest advances in number theory which he (Landau) has lived to see. The theorems themselves belong to the range of ideas of Goldbach's conjecture. But here there is no question of rendering the problem accessible. Important theorems are actually demonstrated.

But the method is not abstruse. Landau says of his own whole exposition (in all essential elements going back to Schnirelmann, as Landau emphasizes) that it could have been written a hundred years ago and can be understood by a reader who knows no differential or integral calculus (to say nothing of functions of a complex variable) and is acquainted only with the exponential function and logarithms.

A fundamental lemma by aid of which the theorems are easily proved is itself established by methods which Brun has used, methods which rest in the last analysis on ideas analogous to those involved in the so-called sieve of Eratosthenes. Besides this a few of the simple results from the analytic theory of prime numbers are needed.

26. *Schnirelmann's theorems.* With the help of algebraic and number-theoretic lemmas due essentially to Brun and by aid of the sieve-process, Landau's exposition passes by way of a proof of the interesting theorem, that the number $f(y)$ of representations of the positive integer y as a sum of two primes has the property that

$$f(y) = O\left(\frac{y}{\log^2 y} \prod_{p|y} \left(1 + \frac{1}{p}\right)\right)$$

where p runs over the prime factors of y, to the demonstration of the following two theorems due to Schnirelmann:

I. If $N(x)$ is the number of integers y less than x each of which is a sum of two primes, then

$$\limsup_{x=\infty} N(x)/x > 0.$$

II. There exists an integer P such that every integer x greater than unity is a sum of at most P primes.

THE PRACTICAL EVALUATION OF RESULTANTS

By T. A. PIERCE, The University of Nebraska

The purpose of the present note is to give a practical method of evaluating the resultant of two equations. The method is particularly effective when the degree of one of the equations is high while that of the other is low.

Use will be made of certain results in the theory of matrices. Let A be a square matrix and let $|A-\lambda I|=f(\lambda)=0$ be the characteristic equation of A. Then if $g(\lambda)=0$ be another equation, the resultant $R(f, g)$ of $f(\lambda)=0$ and $g(\lambda)=0$ is known (Frobenius, Journal für Mathematik, Vol. 84, p. 11) to be

$$R(f, g) = |g(A)| . \tag{1}$$

Furthermore if B is the matrix which has $g(\lambda)=0$ as its characteristic equation then

$$R(g, f) = |f(B)| . \tag{2}$$

The matrix A which has the given equation

$$f(\lambda) = \lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \cdots + a_n = 0 \tag{3}$$

as its characteristic equation is

$$\begin{pmatrix} -a_1, & -a_2, & \cdots, & -a_{n-1}, & -a_n \\ 1, & 0, & \cdots, & 0, & 0 \\ 0, & 1, & \cdots, & 0, & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0, & 0, & \cdots, & 1, & 0 \end{pmatrix}$$

This is readily proved by forming the determinant of this matrix after having subtracted λ from each element of the principal diagonal and expanding this determinant and each resulting determinant according to the elements of their last columns. Similarly the matrix B which has

$$g(\lambda) = \lambda^m + b_1\lambda^{m-1} + \cdots + b_m = 0 \tag{4}$$

as its characteristic equation may be written down.

Since the left members of (1) and (2) differ at most by sign in obtaining the resultant of f and g it is only necessary to calculate that right member of (1) or (2) which is the simpler. It is this consideration of (2) in conjunction with (1) that makes our method practical. The matrices A and B are of order n and m respectively and $g(A)$ and $f(B)$ are matrices of orders n and m. If $m<n$ we compute $|f(B)|$. In forming

$$f(B) = B^n + a_1B^{n-1} + \cdots + a_{n-1}B + a_nI$$

where I is the unit matrix of order m we first reduce this expression by means of

$$g(B) = B^m + b_1B^{m-1} + \cdots + b_{m-1}B + b_mI = 0, \tag{5}$$

since B satisfies its characteristic equation. Every power of B higher than the

$(m-1)$ st in $f(B)$ may be eliminated by using (5). In order to calculate these powers of B some recursion formulas are needed. Let

$$B^r = \alpha_r B^{m-1} + \beta_r B^{m-2} + \cdots + \lambda_r I \tag{6}$$

where $r \geqq m$, then

$$\begin{aligned} B^{r+1} &= \alpha_r B^m + \beta_r B^{m-1} + \cdots + \lambda_r B \\ &= (\beta_r - b_1\alpha_r)B^{m-1} + (\gamma_r - b_2\alpha_r)B^{m-2} + \cdots + (-b_m\alpha_r)I \text{ by (5)} \\ &= \alpha_{r+1}B^{m-1} + \beta_{r+1}B^{m-2} + \cdots + \lambda_{r+1}I \end{aligned}$$

hence $\alpha_{r+1}=\beta_r - b_1\alpha_r$, $\beta_{r+1}=\gamma_r - b_2\alpha_r$, $\cdots$, $\lambda_{r+1}=-b_m\alpha_r$. These are the recursion formulas desired. It is now a simple process to calculate $f(B)$ and evaluate the determinant of the resulting matrix.

As an example we will find the resultant of

$$\begin{aligned} f(\lambda) &= \lambda^{12} + \lambda^7 - \lambda^4 - 2\lambda^3 + 3\lambda + 10 = 0 \\ g(\lambda) &= \lambda^4 - \lambda^2 + 2\lambda - 3 = 0. \end{aligned}$$

The dialytic method of Sylvester would give a determinant of order 16. We have

$$B = \begin{pmatrix} 0, & 1, & -2, & 3 \\ 1, & 0, & 0, & 0 \\ 0, & 1, & 0, & 0 \\ 0, & 0, & 1, & 0 \end{pmatrix}$$

and must first find $f(B)$. To calculate B^{12} and B^7 in terms of B^3, B^2, B, and I we use the recursion formulas

$$\alpha_{r+1} = \beta_r, \quad \beta_{r+1} = \gamma_r + \alpha_r, \quad \gamma_{r+1} = \delta_r - 2\alpha_r, \quad \delta_{r+1} = 3\alpha_r$$

with the initial values $(\alpha_4, \beta_4, \gamma_4, \delta_4) \equiv (0, 1, -2, 3)$. Hence

r	4	5	6	7	8	9	10	11	12
α_r	0	1	−2	4	−4	11	−18	31	−52
β_r	1	−2	4	−4	11	−18	31	−52	100
γ_r	−2	3	−2	7	−14	20	−34	69	−116
δ_r	3	0	3	−6	12	−12	33	−54	93.

Thus $B^{12} = -52B^3+100B^2-116B+93I$ and $B^7 = 4B^3-4B^2+7B-6I$. Substituting these expressions in $f(B)$ we find $f(B) = -50B^3+95B^2-104B+94I$ and this is the matrix

$$\begin{pmatrix} 289, & -494, & 593, & -462 \\ -154, & 289, & -340, & 285 \\ 95, & -154, & 194, & -150 \\ -50, & 95, & -104, & 94 \end{pmatrix}.$$

The value of the desired resultant is the determinant of this matrix and is 584549.

QUESTIONS AND DISCUSSIONS

EDITED BY R. E. GILMAN, Brown University, Providence, Rhode Island

The department of Questions and Discussions in the Monthly is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems which are reserved for the department of Problems and Solutions.

A METHOD OF HANDLING CERTAIN ELEMENTARY INTEGRALS[1]

By OTTO DUNKEL, Washington University

In texts on the calculus the integrals

$$(1)\qquad \int (x^2 + 2Ax + B)^{-1/2}dx, \qquad (2)\qquad \int (x^2 + 2Ax + B)^{1/2}dx,$$

are usually derived from standard types in which $A = 0$. The integration of these standard types is carried out in a variety of ways, and some of the methods involve troublesome computations with the substitutions used. The method which will be here given appears to be simpler in its computations, and it applies just as easily to the general forms above as to the simpler standard types. In both cases (1) and (2) the saving of labor results from the use of the principle called "composition and division" in elementary algebra.

We set

$$(3)\qquad x^2 + 2Ax + B = u^2, \quad (x + A)dx = udu;$$

and there results

$$(4)\qquad \frac{dx}{u} = \frac{du}{x + A} = \frac{d(x + A + u)}{x + A + u},$$

where the last expression in the equality is obtained first by "composition and division," and then by noting that $d(x+u) = d(x+A+u)$. We may now set down immediately the integral of (1)

$$(5)\qquad \int \frac{dx}{u} = \log\,[x + A + (x^2 + 2Ax + B)^{1/2}],$$

where the arbitrary constant is omitted for brevity of writing. This method has been seen in one text, but only for the case $A = 0$ in (1), and there was no extension of the method to (2) in any form. The writer cannot recall the author of the text.

For (2) we use the same transformation (3), and we multiply both sides of the first equation in (4) by u^2 obtaining

$$(6)\qquad udx = \frac{u^2}{x + A}\,du = \frac{x^2 + 2Ax + B}{x + A}\,du.$$

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[1] Read before the sixteenth annual meeting of the Association held at New Orleans, Dec. 31, 1931.

By division there results

$$(7)\qquad \begin{aligned} udx &= (x+A)du + \frac{B-A^2}{x+A}\,du, \\ &= \tfrac{1}{2}\left[udx + (x+A)du + \frac{B-A^2}{x+A}\,du\right], \end{aligned}$$

where the last expression may be seen to result by composition and division, or more simply otherwise. The first two terms in the brackets reduce to $d\{u(x+A)\}$, and the integral of the last term is obvious from (4) and (5). Thus we have for (2)

$$(8)\qquad \int udx = \tfrac{1}{2}(x+A)(x^2+2Ax+B)^{1/2} + \tfrac{1}{2}(B-A^2)\log\left[x+A+(x^2+2Ax+B)^{1/2}\right].$$

It will be observed that the methods above are equivalent in a way to the substitution given in some texts

$$(9)\qquad (x^2+2Ax+B)^{1/2} = u - x.$$

This substitution may be applied to both (1) and (2), but the steps of the process require tedious algebraic and differential computations. In type (1) there is a cancellation of complicated terms and the final result is easily integrated. In the method above the use of "composition and division" avoids the labor wasted in computing terms which finally cancel. When (9) is applied to (2) there are no such serviceable cancellations, and the work becomes so involved that one is led to wish for a less laborious process.

The processes above are obviously inapplicable to (1) when the coefficient of x^2 is -1 instead of $+1$. But we can derive the result for the type corresponding to (2) if we have the integral for the one corresponding to (1). This is not of any practical importance in integration, but we mention it as an interesting fact. Thus

$$\int (B+2Ax-x^2)^{-1/2}dx = \arcsin\left[(B+A^2)^{-1/2}(x-A)\right].$$

Here $B+2Ax-x^2=u^2$, $dx/u=du/(A-x)$, and our integral may be written $\int du/(A-x)$. For the other case we have

$$\begin{aligned} udx &= \frac{u^2}{A-x}\,du = \frac{B+2Ax-x^2}{A-x}\,du = \left(x-A+\frac{B+A^2}{A-x}\right)du, \\ &= \tfrac{1}{2}\left[udx + (x-A)du + \frac{B+A^2}{A-x}\,du\right]. \end{aligned}$$

Hence

$$\int (B+2Ax-x^2)^{1/2}dx = \tfrac{1}{2}(x-A)(B+2Ax-x^2)^{1/2} + \tfrac{1}{2}(B+A^2)\arcsin\left[(B+A^2)^{-1/2}(x-A)\right].$$

Expression for a Determinant in Terms of Five Minors

By E. B. Stouffer, University of Kansas

In 1881 Muir[1] announced two theorems on determinants which he named the Law of Complementaries and the Law of Extensible Minors. These theorems make it possible to write a whole series of relationships among the minors of determinants as soon as a single such relationship is given. In particular, from the expansion of a second order determinant there can be obtained by means of the Law of Extensible Minors a simple expression for a determinant of any order in terms of *five* of its minors. In other words, the value of a determinant is known at once when the values of five properly selected minors are known, even though the order of the determinant is not known and the value of not a single element is known. This fact could hardly have escaped the attention of Muir but there appears to be no specific mention of it in the literature. Consequently, it seems worth while to restate briefly the two laws and to show their application by deriving at least this one interesting result.

Muir states the Law of Complementaries practically as follows: *To every general theorem which takes the form of an identical relationship among a number of the minors of a determinant (the determinant itself may be included as a minor) there corresponds another theorem derivable from the former by substituting for every minor its algebraic complement in the determinant, and then multiplying each term by such a power of the determinant as will make the relationship homogeneous in the elements.*

In order to prove this theorem for a determinant D it is only necessary to write the given relation for the adjoint of D and then replace each minor in this relationship by its equivalent in terms of the algebraic complement of the corresponding minor in D multiplied by the proper power of D.

Muir's statement of the Law of Extensible Minors is essentially as follows: *If any identical relationship be established among a number of minors of a general determinant (the determinant itself may be included as a minor), the minors being denoted by means of their principal diagonals, then a new relationship involving the minors of a determinant with k additional rows and columns is always obtainable by annexing the k new elements in the principal diagonal to the end of the diagonal of every minor occurring in the identity, and then multiplying each term by such a power of the principal minor of order k, consisting of new elements only, as will make the relationship homogeneous in the elements.*

The proof of this theorem consists merely in applying the Law of Complementaries twice, first to the given relationship, considering the minors as belonging to the original determinant, and then to this new relationship, considering the minors as belonging to the extended determinant.

Let us now apply the Law of Extensible Minors to the simplest possible identity

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[1] Transactions of the Royal Society of Edinburgh, vol. 30, pp. 1–4. See also, Muir and Metzler, *A Treatise on the Theory of Determinants*, paragraphs 179 and 187.

$$(1) \qquad \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = |a_{11}a_{22}| = a_{11}a_{22} - a_{12}a_{21}.$$

If we add enough rows and columns to make the order of the determinant n and continue to denote the new elements by a_{ij}, we have by the Law of Extensible Minors

$$(2) \qquad |a_{11}a_{22}a_{33}\cdots a_{nn}| = \frac{|a_{11}a_{33}\cdots a_{nn}|\cdot|a_{22}a_{33}\cdots a_{nn}| - |a_{12}a_{33}\cdots a_{nn}|\cdot|a_{21}a_{33}\cdots a_{nn}|}{|a_{33}\cdots a_{nn}|}.$$

We find thus that *a determinant of order n is a function of four minors of order* $n-1$ *and one of order* $n-2$.

The principal minor $|a_{11}a_{22}|$ has been used as the starting point in the demonstration but it is evident that any principal minor of the second order would serve. However, in applying the theorem to a given determinant it is necessary to observe the single limitation that the minor of order $n-2$ which appears in the denominator of (2) must not be zero. By interchange of rows or columns the determinant can always be put into such a form that the principal minor selected from any fixed $n-2$ rows is different from zero, unless the determinant itself vanishes identically.

Many similar but less simple relationships may be obtained. For example, we may start with a determinant of order three and obtain an expression for a determinant of any higher order in terms of seven minors, three of order $n-1$, three of order $n-2$ and one of order $n-3$. Likewise from the determinant of order three we may obtain an expression for a determinant of any higher order in terms of nine minors of order $n-2$ and one of order $n-3$.

In an earlier paper[1] the author has shown that the Law of Extensible Minors may be used to prove that any determinant can be expressed in terms of fourteen *principal minors*.

RECENT PUBLICATIONS

EDITED BY ROGER A. JOHNSON, Brooklyn College of the City of New York

All books for review should be sent directly to the editor of this department, at Brooklyn College, 66 Court Street, Brooklyn, N. Y., and not to any of the other editors or officers of the Association.

REVIEWS

Applications of the Absolute Differential Calculus. By A. J. McConnell. London, Blackie and Son, 1931. xii+318 pages. 20s.

This excellent work is essentially a textbook on the tensor calculus written primarily for the advanced undergraduate and having as its outstanding char-

[1] This Monthly, vol. 35 (1928), pp. 18–21.

acteristic an abundance of illustrative material. A second distinguishing feature is the large collection of problems, 879 in all, of which forty-one are solved and the remainder left to the student. Some of the latter set may be solved without recourse to paper and pencil while others, although not difficult, furnish considerable additional theory. This book is well written, the notation is modern, and the general plan, which we shall now sketch briefly, is admirable.

Part One is a detailed and elementary exposition of tensor algebra including a treatment of determinants by means of Kronecker deltas and the allied systems. In this and the succeeding section, "Algebraic Geometry," linear transformations are employed almost exclusively.

Under "Algebraic Geometry" we find the application of the preceding theory to cones, conics, and quadrics. Incidentally much of Gibbs' vector analysis appears in somewhat different dress.

The third of the four major subdivisions, "Differential Geometry" is perhaps the most inclusive of all. Here the emphasis has shifted from the linear to the general transformation and from the algebraic to the differential properties of tensors. Some of the topics treated are covariant differentiation, parallel displacement, the Frenet formulas, the Riemann-Christoffel tensor, Gaussian curvature, the first, second, and third ground forms, geodesics, and the principal curvatures.

The last section is devoted to mechanics, electricity, and special relativity. The chapters on mechanics, "Dynamics of a Particle," "Dynamics of Rigid Bodies," and "Mechanics of Continuous Media" presuppose but little preliminary knowledge, whereas the discussion of electricity, for the most part, is not expository, being essentially a tensor reformulation of the fundamental equations. The author has attacked special relativity from the tensor point of view and thereby provided an excellent, though not comprehensive, introduction to the general theory.

Apparently this book contains but few errors. We noted one or two slight misstatements and a half dozen harmless misprints. In addition, there occur, in a few places, what might be considered as errors of omission. For example, in the derivation of the differential equations of the extremal curves no reference is made to the fundamental lemma.

H. V. CRAIG

Debunking Science. By E. T. Bell, University of Washington Chapbooks, #44, 1930, 40 pages.

Scientists, like other men, are constantly overestimating what they know. They repeatedly believe they have proved and disproved more than they have. When a scientific hypothesis works well in mentally organizing a few experimental results and fails only for experiments which are reported in small type or in obscure journals, it becomes a law of nature. A slightly changed cosmology, or a new biological discovery may lead to atheism; while the discovery of a very penetrating ray or an application of a non-definite quadratic form to physics

seems ample justification for concluding that there is a God and that he is a pure mathematician.

It is pleasant, therefore, to review even such a short booklet as this, warning scientists that their group self-respect should prevent them of all people from being overcredulous.

The book is divided into three parts. It lacks a table of contents. I, therefore, furnish it with the following:

Part I. Stop Being Childish and Dishonest—pp. 7–17. (This part is a pleasant vituperation against credulity and is very good reading, as is always true with whole-hearted scolding when there is a vein of humor running through it. See Stubb's exhortations to his crew in Moby Dick. Both the irony and the pleasantry are well done.)

> Science as a pseudo religion and religion as a pseudo science. As Disko Troop said, "Keep things separate," 7; The flapper as the sculptured goddess of Debunkiana, 9; Bunk in science more disgusting than elsewhere, 11; Blessed be the doubters, 14; Science worth saving, 16.

Part II. Even Mathematics Needs Debunking—pp. 17–24. (This part is a little more solemn. Toe treading isn't so much fun among friends. This chapter, however, gives a very good account of the highly optimistic attitude of pre-Brouwer Mathematics.)

> Real start of modern debunking in mathematics dates from 1908, 17; Popular expositors aren't aware of it yet, 17; What is proof and what is nonsense is not yet settled, 18; The pleasant pre-Brouwer world, 19; Not even mathematics has reached certainty, 23.

Part III. Description of the Mathematical Controversy—pp. 24–40. (This is really a very good non-technical hint of what is going on in mathematics. It is about all that can accurately be said to the non-mathematician. Though well written, it will never be included in the "World's Greatest Humor.")

> Kronecker started it, 24; The tender-minded and tough-minded, 25; A paradox, 27; Berkeley wins, 29; Cantor and Dedekind, 29; Kronecker, 30; Brouwer, 32; Russell's Principia stillborn, 34; Hilbert, 36; Peroration, and final hymn without benediction, 40.

Part IV. Not written. (Should be technical discussions for the Transactions.)

Part V. Not written. (By extrapolation it would be so solemn that it should be buried in an Academy publication.)

Really the pamphlet is worth reading, and the author's point of view needs emphasis. It is a little too bad that the author had such a good time writing Part I that he couldn't keep mad for the next two parts.

MARK H. INGRAHAM

Enciclopedia delle Matematiche Elementari a cura di L. Berzolari, G. Vivanti e D. Gigli. Volume 1, part 2. Milan, Ulrico Hoepli, 1932. XVI+611 pages.

The fact that there is now no modern mathematical encyclopedia in the English language is perhaps sufficient evidence of the magnitude of such an un-

dertaking even if it is restricted to the elements of our subject. Part I of the present volume appeared in 1930 and was reviewed in this MONTHLY, volume 37 (1930), page 378, by Professor A. A. Bennett. The second and somewhat larger part of this volume is dated 1932 and completes the treatment of the subject of analysis in a little over a thousand pages, while the *Encyklopädie der mathematischen Wissenchaften* devotes a little over five thousand pages to the same subject. The treatment in the present volume is however more extensive than that given of the same subject in the *Encyklopädie der Elementar-Mathematik* by Weber and Wellstein.

The thirteen main headings of this second part, with the number of pages devoted to each of these subjects and the authors of the articles, are as follows: combinatory calculus (9), L. Berzolari; elements of the theory of groups (51), L. Berzolari; determinants (30), L. Berzolari; linear equations (13), L. Berzolari; linear substitutions and linear, bilinear and quadratic forms (28), L. Berzolari; rational functions of one or more variables (37), O. Nicoletti; general properties of algebraic equations (59), O. Nicoletti; equations of the second, third and fourth degree and other particular algebraic equations, systems of algebraic equations of elementary type (57), E. G. Togliatti; methods for the discussion of problems of the second degree and remarks on some of the third and fourth degree (63), R. Marcolongo; limits, series, continued fractions and infinite products (45), G. Vitali; elements of infinitesimal analysis (101), G. Vivanti, relations between the theory of aggregates and elementary mathematics (11), G. Vivanti; the analytic functions from an elementary point of view (29), S. Pincherle.

The names of the authors of these various articles are a sufficient guarantee of the high standards maintained in the preparation of this very useful work, which was under consideration for more than twenty years but was greatly delayed by the world war and its adverse consequences. From the titles noted in the preceding paragraph it is clear that the term elementary mathematics is used here with a much wider meaning than the one found, for instances, in the *New International Encyclopedia*, 1923, under the term "Mathematics," where it is stated that "the period of the development of elementary mathematics closes with the seventeenth century." Such a subject as the theory of groups, in particular, was practically unknown in 1700 but its elements are developed in the present encyclopedia in an unusually clear and comprehensive manner, together with a large number of references to sources.

One of the most valuable features of this encyclopedia is its large number of references to the earliest developments, and hence it is very useful to those who are mainly interested in the history of elementary mathematics. Its object is to give proofs of a number of the most fundamental theorems and simply to give references to the places where proofs of others can be easily found, with an emphasis on the earliest ones. In a few cases one misses here the most recent historical developments relating to the subjects which are discussed. For instance, no references appear therein to the recent discoveries relating to the early

partial solutions of quadratic equations by the ancient Babylonians, which were published by O. Neugebauer and others in the recently inaugurated periodical entitled *Quellen und Studien zur Geschichte der Mathematik*. In fact, this periodical is not included in the lists which appear, together with their abbreviations, near the beginning of each of the two parts of the volume under consideration.

The author index of the present part includes about 800 different names. In view of the emphasis on the earliest developments one would not expect that many of these are those of American mathematicians except when the subjects treated are modern as in the case of the theory of groups. The number of the different names of American mathematicians therein is however more than forty. An international spirit is also exhibited by the fact that the largest numbers of references under individual names appear under those of the two non-Italian mathematicians, A. L. Cauchy and L. Euler. Unfortunately the volume does not contain a subject index, but it is to be hoped that such an index will appear in a later volume of this work. The arrangement of the material is topical under the main headings noted above, and the treatment aims to develop only the more elementary concepts, and hence it is especially adapted to the needs of those who desire to restrict their attention for the time being to the most fundamental ideas involved in a particular subject relating to elementary mathematics.

According to the announced plan the completed work will be composed of three volumes. Volume II is to be devoted to geometry while volume III is expected to treat a variety of different subjects including the history and the teaching of mathematics. According to the original plan these two subjects were also to be treated in the closing volume of the *Encyklopädie der mathematischen Wissenschaften* but up to the present time nothing thereon has appeared as a part of this work. Hence the Italian encyclopedia under consideration has no modern forerunner along these lines and it will be interesting to see what advances will be made along these difficult and backward lines of mathematical development where permanent progress seems to be most uncertain. In view of the remarkable recent advances made in geometry by Italian mathematicians the volume on geometry will be awaited with high expectations.

On page 43 it is stated that the natural numbers form a group when they are combined according to multiplication while on page 33 it is noted that these numbers do not constitute a group if we assume as a postulate with respect to infinite groups a very useful property of finite groups. It seems to the reviewer very unfortunate that the term group when it relates to an infinite number of elements is not fully defined in the present work, for it would be very desirable to establish uniformity of usage along this line in the modern mathematical literature. On page 31 it is stated that the octic group is generated by three distinct transpositions which is obviously impossible. The number of such oversights is, however, small; the work as a whole can be heartily recommended and it should do much to secure higher standards as regards a knowledge of elementary

mathematics. In view of the fact that the mathematical language is so largely international those who know very little Italian will be able to consult this work without much language trouble and by doing so they may be able to improve their knowledge of a language which has become essential to all who desire to keep in touch with the most important modern mathematical developments.

G. A. MILLER

Intermediate Calculus. By Percey F. Smith and William R. Longley. Boston, Ginn and Company, 1931. xiii+457 pages. $3.20.

There are a few colleges and universities in this country in which some or all of the entering students have completed the study of trigonometry in their preparatory schools. For such, a brief course in analytics followed by one in the calculus is the usual procedure. Or the course may be built around the calculus with the analytical geometry developed as needed. Thus at Yale University we find written for this particular purpose Longley and Wilson's *An Introduction to The Calculus*. It is not desirable to follow such a course with the ordinary text on the elementary calculus, hence Smith and Longley have prepared this book entitled *Intermediate Calculus*. In reviewing it we should keep in mind, first the contents of Longley and Wilson, since the two books together constitute a unit; secondly, we should remember that in 1929 Smith and Longley published a revision of Granville's Calculus, with which we can compare the present work.

Following a collection of formulas in Chapter I, there are three review chapters—one of fourteen pages mainly on the analytical geometry of conic sections, one of eighteen pages on the differential calculus of algebraic functions and one of thirty-eight pages on the integral calculus of algebraic functions. Since the technique of differentiation and integration was learned in the previous course, these fifty-six pages, except for a few definitions, are devoted exclusively to problems in maxima and minima, curve tracing, rates, areas, volumes of solids of revolution, moments of inertia, work, and fluid pressure. The trapezoidal rule and Simpson's rule for the approximate evaluation of the definite integral are given. Here in compact form the student has recalled to him all that is absolutely essential for his study of this book. It is well done; the only slight criticism would be the use of dy/dx as the symbol for the derivative.

The transcendental functions are introduced in Chapters V and VI, and treated fully as new material with applications to curve tracing, simple harmonic motion, damped and forced vibration. In Chapters VII and VIII are the usual methods of integrating expressions which do not come directly under the standard forms. Throughout these four chapters are numerous exercises which enable the student to develop the technique of differentiation and integration of transcendal functions.

In Longley and Wilson there was no discussion of polar coordinates and but brief mention of parametric equations. Consequently these topics must be developed from the very beginning; curves must be plotted, equations of cycloidal

curves must be derived, and the application of the calculus pointed out. Various graphic methods as well as Newton's method of calculating the approximate value of the roots of algebraic and transcental equations are discussed. A chapter on empirical equations fits a straight line to the given points by the *method of averages*. Six other two-constant equations, which can be handled similarly to that of the straight line, are treated together with one three-constant curve, the general parabola $y=a+bx+cx^2$. While the mathematical derivation of the *method of least squares* is beyond the scope of the book, one regrets that in a mathematical text-book this method could not have been stated and used instead of the *method of averages*.

The chapter on curvature is practically the same, text and exercises, as the one in Granville, Smith and Longley. The paragraphs on transformation of derivatives and on involutes have been omitted, and more added on the tangential and normal components of the acceleration vector in curvilinear motion. The same is true for the chapter on ordinary differential equations. The discussion of linear differential equations of the nth order with constant coefficients is not in the later book. The exercises are the same; in some y'' and y' are used instead of d^2y/dx^2 and dy/dx respectively. The only multiple integrals treated are the double integrals in the final chapter of the book; this again is precisely as found in the Smith and Longley revision of Granville. The chapters on series and on partial differential equations differ in arrangement of subject matter and in problems from those in the other book. The material is condensed, especially that on partial differential equations, and some of the applications omitted. A chapter on solid analytical geometry is inserted before that on partial differential equations. In the chapter on series the reviewer feels that it should be stated that an infinite series represents the function only when the remainder approaches zero, as the number of terms increases without limit.

The dimensions of the book make it more convenient to use than the 1929 revision of the Granville text. The figures are good, the printing well done. A short table of integrals has been placed at the end of the book. Since in so many colleges it is necessary to teach trigonometry and even algebra in the first year, many teachers will be prevented from utilizing this as a text-book.

J. R. MUSSELMAN

Elements of the Theory of Integers. By Joseph Bowden: Revised Edition. Published by the Author, Garden City, N. Y., 1931. 268 pages. $2.50.

This is substantially a reprint of the earlier (1903) edition, save for some changes in the last chapter. The work forsakes tradition in many superficial ways. Such spelling as "giv," "ar," "hav," "ther," "orderd," first suggest an attitude that one meets throughout. There are correspondingly new terms employed, and new symbols for old terms. Probably few will recognize the relations, "sul," "mul," "kor," the inverted Pi for "is prime to" and many others.

The work deals essentially with the rational integers and their elementary properties, with emphasis upon the logical aspects. It is only in the last chapter, that dealing with linear congruences, that any mathematical relations entirely new to the high school pupil are touched upon; but the extensive use of axioms, starting even with axioms for the formal treatment of hypothetical propositions and the explicit discussion of the Aristotelian syllogism, will be unfamiliar (and perhaps unwelcome). Probably the most interesting item from the young reader's point of view is the explicit use of single symbols for the numbers, 10, 11, $\cdots$, 16, thus liberating one from the accidental dependence upon the decadic notation appropriate to the decimal system, although little is made of this opportunity.

The book has been employed as the text in a course for first year students and is apparently intended for this purpose, so should probably be considered with the purpose in mind. Indeed except for notation there are no new concepts suggested that would recommend the work for advanced students. There are no exercises! The references are numerous but may be viewed as rather in the nature of acknowledgment of sources than as suggestions for the encouragement of youthful readers. Since no numbers other than the traditional ones of elementary arithmetic are suggested, the reader is likely to exclaim, "Why all this talk about prefactor, postfactor, etc. etc." No proofs of consistency of independence are offered so that the discussion of these concepts although verbally clear must remain almost without significance. The choice of logical axioms (resulting in a system neither sufficient nor independent) does not encourage further study.

With the wealth of mathematics available which is of importance and adapted to arouse the interest of the immature freshman, one may question whether a course based upon this text may not do harm in many cases by strengthening the prejudice of the student to the effect that beyond high school topics there is no mathematics worthy of attention.

The work is in general careful and accurate and the author is well-informed upon traditional matters. Whether the project is psychologically adapted to the training of the young is the only question at issue. With essentially the same commendable purpose in mind a markedly contrasting work may be mentioned, contrasting as to the type of readers addressed, the choice of material, the style, the contacts with other fields, and particularly the success of the venture, namely H. Beck: Einführung in die Axiomatik der Mathematik, which for its purpose one may recommend unreservedly.

A. A. BENNETT

A LETTER TO THE EDITOR

DEAR SIR:

I was surprised to read in the December number of the MONTHLY in the review of Smail's algebra, "Here is something really new: a textbook which does

not present the subject matter of college algebra as a sequence of disconnected topics, etc." Evidently the reviewer, who claims more for the book than its author, does not take his Ecclesiastes seriously for there I read: "Is there anything whereof it may be said, See, this is new? It hath been already of old time, which was before us." In support of the Preacher, I rise to remark that fifteen years ago Wilczynski wrote and Slaught edited a college algebra,[1] conceived in the same spirit and executed on the same general pattern as Smail's, and containing some of the topics which the reviewer noted as omitted in the latter book. I don't know how widely Wilczynski's book was adopted—though we used it at Washington for several years—but it seems strange that a work of two such prominent mathematicians should have been overlooked.

Do mathematics and pedagogy move so fast that a book is swallowed up in oblivion in 15 years? The Preacher scores again when he observes: "One generation passeth away and another generation cometh . . . There is no remembrance of former things." Perhaps the answer is "Of making many books there is no end." At any rate it will be interesting to see which shall prevail—the reviewer's hope that Smail's book will prove popular and successful, or the Preacher's prophecy, given another fifteen years, "neither shall there be any remembrance of things that are to come with those that shall come after."

Incidently, I may add that we are adopting Smail's book.

Sincerely,

R. M. WINGER

University of Washington,
January 6, 1932.

PROBLEMS AND SOLUTIONS

EDITED *by* B. F. FINKEL, OTTO DUNKEL *and* H. L. OLSON

Send all communications about Problems and Solutions to B. F. Finkel, Springfield, Mo. All manuscript should be typewritten, with double spacing, and with margins at least one inch wide.

PROBLEMS FOR SOLUTION

N.B. Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks or results found in readily accessible sources, will not be proposed as problems for solution in the MONTHLY. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

3535. *Proposed by J. M. Feld, Brooklyn College of the City of New York.*

The pedal curve p with respect to a point P of a given curve c is defined as the locus of the feet of the normals from P to the tangents of c, and c is known

[1] Reviewed in the Monthly, Vol. 24 (1917), p. 230.

as the negative pedal of p. Prove that the negative pedals of circles and straight lines with respect to a point P, not on the circles or straight lines, are respectively central conics and parabolas having a focus at P. The parabolas are tangent to their pedals and the central conics are doubly tangent to theirs.

3536. *Proposed by Martin Rosenman, Brooklyn, New York.*

Consider fractions of the form $1/2, 1/3, 1/4, 1/5, \cdots$. We seek to determine which n of these fractions (repetitions allowed) give a sum as near unity as possible but actually less than it. Thus for $n=3$, we have $1/2+1/3+1/7=41/42$. Prove or disprove that, in general, the first n of the fractions in the series $1/2+1/3+1/7+1/43+1/1807\cdots$ give the desired result; in which series each denominator exceeds by 1 the product of all preceding.

3537. *Proposed by W. H. Rasche, Virginia Polytechnic Institute.*

The letters H, V, and P designate a system of mutually perpendicular coordinate planes, and the letters α, β, and γ the angles which a plane Q makes with H, V, and P respectively. If none of these angles exceeds 90°, what are the extreme limits of the sum $\alpha+\beta+\gamma$?

3538. *Proposed by Eugene M. Berry, Lynchburg College.*

Solve the differential equation

$$ldy = F(x)\ \sin\ (x-y)dx,$$

where l is a constant and $F(x)$ is any function of x.

3539. *Proposed by R. E. Gaines, University of Richmond.*

A slender rod of length $2a$ rests on a circular table of radius r, $r>a$. What are the probabilities that neither end, one end, or both ends, will project over the edge of the table?

3540. *Proposed by Nathan Altshiller-Court, University of Oklahoma.*

Consider the four spheres having for diameters the four lines joining the ends of two diameters, chosen arbitrarily, in two given circles of a given sphere. Prove that the radical center of the four spheres is fixed.

3541. *Proposed by J. B. Reynolds, Lehigh University.*

A rod of length $2a$ standing on a rough level plane of coefficient of friction μ falls from a vertical position. At what angle with the vertical will the rod begin to slip at the bottom?

3542. *Proposed by W. E. Buker, Leetsdale, Pennsylvania.*

It is known that $3^2+4^2=5^2$ and $3^3+4^3+5^3=6^3$. Find other integral values of a, r, and l, if they exist, such that

$$a^r + (a+1)^r + (a+2)^r + \cdots + (a+l)^r = (a+l+1)^r$$

SOLUTIONS

3408 [1930, 38]. *Proposed by J. V. Uspensky.*

Show that the integral

$$V_n = \int_0^1 \int_0^1 \cdots \int_0^1 \frac{x_1^2 + x_2^2 + \cdots + x_n^2}{x_1 + x_2 + \cdots + x_n} dx_1 dx_2 \cdots dx_n$$

converges to the limit $2/3$ when n increases indefinitely and that the product $n(V_n - 2/3)$ remains bounded.

Solution by the Proposer.

By the use of the formula

$$(1) \qquad (x_1 + x_2 + \cdots + x_n)^{-1} = \int_0^\infty e^{-t(x_1+x_2+\ldots+x_n)} dt,$$

the integral denoted by V_n may be reduced to

$$(2) \qquad V_n = n \int_0^\infty [\psi(t)]^{n-1} \phi(t) dt, \quad \psi(t) = (1 - e^{-t})t^{-1}, \quad \phi(t) = \int_0^1 e^{-t\xi^2} d\xi.$$

The whole question is now reduced to finding an asymptotic expression for the integral V_n in (2). To this end it is to be observed that $\psi(t)$ is a function of t which decreases from 1 to 0 as t varies from 0 to ∞. Hence the equation

$$(3) \qquad \psi(t) = (1 - e^{-t})t^{-1} = 1 - y$$

has one and only one solution in t for each value of y, $0 < y < 1$. This solution can be expressed as a power series in y

$$(4) \qquad t = 2y + \frac{4}{3} y^2 + \cdots,$$

which is convergent for $y \leqq c$, c being any positive number less than unity. For the chosen value of c, we suppose that t has the value a. Hence

$$(5) \qquad dt = [2 + y\theta(y)]dy,$$

where $|\theta(y)| < K$, K being a suitably chosen constant for the interval $0 \leqq y \leqq c$, and where $0 \leqq t \leqq a$.

It will now be convenient to split the integral in (2) into two parts as follows:

$$(6) \qquad I_1 = \int_a^\infty [\psi(t)]^{n-1} \phi(t) dt, \quad I_2 = \int_0^a [\psi(t)]^{n-1} \phi(t) dt,$$

and we consider first I_1. We have

$$\phi(t) < \int_0^1 e^{-t\xi} d\xi = \psi(t) \leqq \psi(a) = h < 1.$$

Hence

$$I_1 < \int_a^\infty [\psi(t)]^n \, dt < h^{n-2} \int_a^\infty t^{-2} dt = h^{n-2} a^{-1},$$

(7)
$$I_1 = \beta h^{n-2} a^{-1}, \quad 0 < \beta < 1.$$

For the part I_2 we observe that we may write

$$\phi(t) = \tfrac{1}{3} - t \int_0^1 e^{-\theta t \xi^3} d\xi = \tfrac{1}{3} - \omega t, \quad 0 < \omega < \tfrac{1}{4}.$$

Hence

$$I_2 = \tfrac{1}{3} \int_0^a [\psi(t)]^{n-1} dt - \omega^1 \int_0^a [\psi(t)]^{n-1} t dt, \quad 0 < \omega^1 < \tfrac{1}{4}.$$

We now make a change of variable by means of (3) and (5), and for brevity we write

$$f(y) = [2 + \frac{4}{3} y + \cdots][2 + \theta(y)].$$

Then

$$\begin{aligned} I_2 &= \tfrac{1}{3} \int_0^c (1 - y)^{n-1} [2 + y\theta(y)] dy - \omega^1 \int_0^c (1 - y)^{n-1} y f(y) dy \\ &= \tfrac{2}{3} \int_0^c (1 - y)^{n-1} dy + H \int_0^c (1 - y)^{n-1} y dy \\ &= \tfrac{2}{3} n^{-1} - \tfrac{2}{3}(1 - c)^n n^{-1} + H n^{-1} (n + 1)^{-1} [1 - (nc + 1)(1 - c)^n]. \end{aligned}$$

Combining the results and introducing certain simplifications we have

$$V_n = \tfrac{2}{3} - \tfrac{2}{3}(1 - c)^n + \frac{\alpha}{n} + \frac{\beta h^{n-2}}{a};$$

and from this expression for V_n results the statement in the problem.

Also solved by Elbert F. Cox.

3460 [1930, 508]. *Proposed by B. F. Kimball, University of New Hampshire.*

Show that the integral:

$$V_n = \int_0^1 \int_0^1 \cdots \int_0^1 \frac{x_1^r + x_2^r + \cdots + x_n^r}{x_1 + x_2 + \cdots + x_n} dx_1 dx_2 \cdots dx_n$$

converges to the limit $2(r+1)^{-1}$ when n increases without limit, and that the product $n[V_n - 2(r+1)^{-1}]$ remains bounded; r to be any real number greater than -1. A generalization of 3408 [1930, 38].

A Note by Otto Dunkel: The solution of this problem follows easily along the lines given in the interesting solution of 3408. Since here the integrand is in-

finite when r is negative, and since an additional integration will be introduced with an infinite limit and the order of the integration will then be changed, it will be shown that the integral has a finite value which is unaltered by these changes. The given integral may be divided into n integrals of which the ith integral is the limit of

$$I_\epsilon = \int_0^1 \cdots \int_\epsilon^1 \cdots \int_0^1 \frac{x_i^r}{x_1 + x_2 + \cdots + x_n} dx_1 dx_2 \cdots dx_n,$$

as ϵ approaches zero, where ϵ is the lower limit of x_i. Since the integrand for I_ϵ is continuous we may change the order of integration and write

$$I_\epsilon = \int_\epsilon^1 \int_0^1 \cdots \int_0^1 \int_0^\infty \xi^r e^{-t(x_1+x_2+\cdots x_{n-1}+\xi)} dt dx_1 \cdots dx_{n-1} d\xi.$$

The integral with respect to t is split into two parts, the first of which has the limits 0 and l, and the second part has the limits l and ∞. For the second part we have

$$\int_l^\infty e^{-t(x_1+x_2+\ldots+\xi} dt < \int_l^\infty e^{-lt} dt = \epsilon^{-1} e^{-lt}.$$

For a given ϵ we can find a number L so that for an arbitrarily small η we have $\epsilon^{-1}e^{-lt} < (1+r)\eta$ when $l \geqq L$. There are now two parts for I_ϵ, and in the first part we may change the order so that we have

$$I_\epsilon = \int_0^l \int_\epsilon^1 \int_0^1 \cdots \int_0^1 \xi^r e^{-t(x_1+x_2+\ldots+\xi)} dx_1 dx_2 \cdots dx_{n-1} d\xi dt + \eta', \eta' < \eta, l \geqq L.$$

Thus

$$I_\epsilon = \int_0^\infty \left[\int_\epsilon^1 \xi^r e^{-t\xi} d\xi \right] \left[\int_0^1 e^{-tx} dx \right]^{n-1} dt.$$

The fact that I_ϵ approaches a limit I for $\epsilon \to 0$ results from the rest of the proof. Thus

$$V_n = nI, \quad I = \int_0^\infty \phi(t) [\psi(t)]^{n-1} dt,$$

$$\phi(t) = \int_0^1 \xi^r e^{-t\xi} d\xi, \quad \psi(t) = (1 - e^{-t}) t^{-1}.$$

For the case where r is not negative the proof in 3408 requires merely a change of the exponent 2 to r. For the case of r negative, $1+r>0$, we introduce a trifling change in the part I_1. Here we have $\phi(t) \leqq (1+r)^{-1}$. Then in the same notation we have

$$I_1 = \int_a^\infty \phi(t)[\psi(t)]^{n-1}dt < (1+r)^{-1}\int_a^\infty [\psi(t)]^{n-1}dt$$

$$< [\psi(a)]^{n-3}(1+r)^{-1}\int_a^\infty t^{-2}dt = h^{n-3}(1+r)^{-1}a^{-1}$$

$$I_1 = \beta(1+r)^{-1}a^{-1}h^{n-3},\ \beta < 1.$$

The rest of the proof follows as in 3408.

3480 [1931, 170]. *Proposed by G. W. Wishard, Norwood, Ohio.*.

Prove the following theorem: *If 1 be annexed to any triangular number in the nonary scale of notation, the result will be another triangular number.* Thus, 1, 11 111, 1111, etc. *ad infinitum* are triangular numbers in the nonary system.

Solution by Helen A. Merrill, Wellesley College.

To annex 1 to any number in the nonary scale of notation is to multiply it by 9 and add 1. Any triangular number is of the form $\frac{1}{2}n(n+1)$. Then $9\cdot\frac{1}{2}n(n+1)+1=\frac{1}{2}(3n+1)(3n+2)$, a triangular number. Q. E. D.

If a triangular number written to the base 3 has 01 annexed to it, the resulting number is triangular. Similarly, if a triangular number written to the base 25 has 3 annexed, or if written to the base 5 has 03 annexed, the resulting number is triangular. Or, more generally, if a triangular number written to the base $(2k+1)^2$ has the digit $\frac{1}{2}k(k+1)$ annexed; or, written to the base $2k+1$, $k\leqq 3$, has the digits 0 and $\frac{1}{2}k(k+1)$ annexed; or written to the base $2k+1$, $k>3$, has the two digits representing the number $\frac{1}{2}k(k+1)$ annexed; the resulting number is triangular.

Also solved by H. T. R. Aude, J. E. Burnam, Mannis Charosh, W. R. Church, Ralph Deutsch, L. S. Johnston, Emma T. Lehmer, A. Pelletier, R. C. Staley, Emory P. Starke, and the Proposer.

3485. [1931, 171]. *Proposed by J. M. Feld, New York, N. Y.*

Let the sides of triangle $A_1A_2A_3$ be trisected. A_{ij} is the trisection point on a_j nearer A_i. Let P_i be the intersection of $A_{ij}A_j$ and $A_{ik}A_k$ and let Q_i be the intersection $A_{kj}A_j$ and $A_{jk}A_k$. Prove:

(1) The triangles $P_1P_2P_3$ and $Q_1Q_2Q_3$ are homothetic to $A_1A_2A_3$ and the common homothetic center is the centroid of $A_1A_2A_3$.

(2) $$P_1P_2/A_1A_2 = \tfrac{1}{4},\ Q_1Q_2/A_1A_2 = \tfrac{1}{3}.$$

(3) If C is the intersection of $A_{23}A_3$ and $A_{12}A_2$, and D is the intersection of $A_{13}A_3$ and $A_{32}A_2$, CD is parallel to A_2A_3.

Solution by J. R. Musselman, Western Reserve University.

Given any triangle $A_1A_2A_3$; choose the axes of the coordinate system so that the coordinates of the vertices shall be $A_1:(0, b)$; $A_2:(-a, 0)$; $A_3:(c, 0)$. Easy

calculation will furnish the coordinates of the points P_i and Q_i as follows:

$$P_1: [(c - a)/4, 2b/4]; \quad Q_1: [2(c - a)/5, b/5]$$
$$P_2: [(c - 2a)/4, b/4]; \quad Q_2: [(2c - a)/5, 2b/5]$$
$$P_3: [(2c - a)/4, b/4]; \quad Q_3: [(c - 2a)/5, 2b/5].$$

It is now an easy matter to prove that the lines P_iQ_i are the medians of the triangle $A_1A_2A_3$ and therefore (1) that $P_1P_2P_3$ and $Q_1Q_2Q_3$ are homothetic with the centroid G as the center of similitude. Also (2) one can prove that the sides of the triangles $P_1P_2P_3$ and $Q_1Q_2Q_3$ are parallel respectively to the sides of $A_1A_2A_3$, and a calculation of their lengths proves that

$$P_1P_2/A_1A_2 = \tfrac{1}{4}, \; Q_1Q_2/A_1A_2 = \tfrac{1}{5}.$$

It is worth while noticing that the triangles $P_1P_2P_3$ and $Q_1Q_2Q_3$ are fourfold perspective; the four centers of perspection are the three vertices $A_1A_2A_3$ and the centroid G. Here we have a simple example of two fourfold perspective triangles.

(3) The trisectors meet in six other points whose coordinates are

$$C_1: [(c - 4a)/7, 2b/7]; \quad D_1: [(4c - a)/7, 2b/7]$$
$$C_2: [(2c - a)/7, 4b/7]; \quad D_2: [(c - 2a)/7, 4b/7]$$
$$C_3: [2(2c - a)/7, b/7]; \quad D_3: [2(c - 2a)/7, b/7].$$

Hence, we see that the three lines C_iD_i are parallel to A_2A_3 and equal in length respectively to 3/7, 1/7, 2/7 of A_2A_3; that the three lines C_2D_3, C_3D_1 and C_1D_2 are parallel and equal respectively to 3/7, 1/7, 2/7 of A_1A_2; and that the three lines C_3D_2, C_1D_3, C_2D_1 are parallel and equal respectively to 3/7, 1/7, 2/7 of A_1A_3. The triangles $C_1C_2C_3$ and $D_1D_2D_3$ are threefold perspective, the three axes of perspection are the medians of the triangle $A_1A_2A_3$ which meet at G. The three centers of perspection are on the line at infinity. The six points C_1, C_2, C_3, and D_1, D_2, D_3 lie on an ellipse whose center is G.

A Note by the Editors: All of these results may be deduced from the similar triangles of the figure.

Also solved by J. W. Clawson, Ralph Deutsch, L. S. Johnston, A. Pelletier, F. Underwood, and the Proposer.

NOTES AND NEWS

Readers are invited to contribute to the general interest of this department by sending items to Professor H. W. Kuhn, Ohio State University, Columbus, Ohio.

INTERNATIONAL CONGRESS OF MATHEMATICIANS
ZURICH, SWITZERLAND,
SEPT. 4–12, 1932

The International Congress of Mathematicians will be held in the Federal Institute of Technology at Zurich September 4–12 next. The scientific part of

the program will consist of a large number of general lectures and section meetings. The lectures are intended to provide a complete survey of the present situation in mathematics. Eminent lecturers have, of course, been secured.

The section meetings are intended for briefer discussions of the results of new investigations. The following sections are planned: 1. algebra and theory of numbers; 2. analysis; 3. geometry; 4. probability and insurance mathematics; 5. astronomy; 6. mechanics and mathematical physics; 7. mathematical-technical science; 8. philosophy; 9. history; 10. pedagogy.

In addition to these scientific meetings, social gatherings and excursions to different points of interest in Switzerland, are being planned, so that the mathematicians' sojourn at Zurich will be made pleasant and interesting in every respect. Mathematicians from all over the world are cordially invited to participate in this event.

The Edison medal for 1931 has been awarded to E. W. Rice, Jr., of the General Electric Company, "for his contributions to the development of electrical systems and apparatus and his encouragement of scientific research in industry."

Professor Warren Weaver, Chairman of the Department of Mathematics at the University of Wisconsin, was appointed Director of the Division of Natural Sciences of the Rockefeller Foundation, beginning February 1, 1932.

Professor T. R. Hollcroft, of Wells College, delivered an address entitled *Algebraic curves and surfaces with assigned singularities* before the mathematical section of the University of Durham Philosophical Society, at Armstrong College, Newcastle, on November 27, 1931.

Professor J. W. Young, of Dartmouth College, a charter member and recent President of the Association, died at his home in Hanover, N. H., on February 17, 1932.

The following courses in mathematics are announced for the summer 1932:

University of Chicago, first term, June 20 to July 22; second term, July 25 to August 26. In addition to the regular courses in differential and integral calculus, the following advanced courses will be offered: By Professor H. E. Slaught: Theory of definite integrals. By Professor L. E. Dickson: Theory of numbers; Algebras and their arithmetics. By Professor E. P. Lane: Analytic mechanics; Surfaces and congruences. By Professor L. M. Graves: Solid analytic geometry; Theory of functions of a complex variable. By Professor W. T. Reid: Differential equations. By Professor Wilhelm Blaschke: Introduction to differential geometry; Topological questions of differential geometry. By Professor G. Bol: Galois Theory of equations. By Professor Walter Bartky: Partial differential equations in mathematical physics; Modern theories of differential equations II.

Columbia University, July 6 to August 14. In addition to courses in trigo-

nometry, solid geometry, analytic geometry, calculus, and methods of teaching secondary mathematics, the following advanced courses are offered: By Professor E. Kasner: Introduction to modern mathematics; Differential geometry. By Professor W. B. Fite: Functions of a complex variable. By Professor J. F. Ritt: Differential equations. By Professor R. G. Archibald: Theory of numbers.

Cornell University, July 11 to August 19. In addition to the usual elementary work, the following advanced courses will be offered: By Professor Virgil Snyder: Algebraic plane curves; Teachers' Course. By Professor F. R. Sharpe: Projective geometry. By Professor W. A. Hurwitz: Advanced calculus. By Professor W. B. Carver: Theory of numbers. By Professor D. C. Gillespie: Functions of a complex variable. Reading and research work will be directed by Professors J. I. Hutchinson, Virgil Snyder, F. R. Sharpe, W. A. Hurwitz, W. B. Carver, D. C. Gillespie, C. F. Craig.

University of Illinois, June 20 to August 13. In addition to the usual courses in college algebra, trigonometry, analytic geometry, and calculus, the following advanced courses are offered: By Professor A. B. Coble: Elliptic functions; Introduction to higher algebra. By Professor Arnold Emch: Geometric transformations; Geometry. By Associate Professor A. R. Crathorne: Analysis; Calculus of variations. By Assistant Professor H. R. Brahana: Algebra. By Assistant Professor H. W. Bailey: Fundamental concepts of mathematics. By Dr. L. L. Steimley: Introduction to higher analysis. By Dr. V. A. Hoersch: Introduction to higher geometry.

University of Iowa, first term, June 13 to July 21. In addition to courses in college algebra, trigonometry, analytic geometry and calculus, the following subjects are offered: By Miss Ruth Lane: Subject-matter and teaching of mathematics. By Professor Reilly: Quadrature and cubature formulas; Seminar in interpolation. By Professor Dines: Linear inequalities; Seminar in analysis. By Associate Professor Wylie: Celestial mechanics; Astronomy; Mathematics of finance. By Associate Professor Woods: Elliptic integrals; Modern geometry. By Assistant Professor Ward: Differential equations; Tensor analysis. By Dr. Robinson: Advanced calculus. By Mr. Fischer: Statistics. By the Staff: Reading and Research. Second term, July 25 to August 25. By Professor Chittenden: Advanced calculus; Theory of sets of points; Seminar in topology. By Professor Nordgaard: The history of mathematics; Theory of equations. By Dr. Conkwright: Differential equations; Theory of numbers. By Dr. Craig: Theory of probability. By the Staff: Reading and Research.

Johns Hopkins University, June 27 to August 5. By Professor F. D. Murnaghan: College algebra; Analytic geometry; Tensor analysis.

University of Kansas, June 8 to August 3. In addition to courses in trigonometry, algebra, and calculus, the following advanced courses are offered: By Professor Smith: Modern analytical geometry; Higher plane curves; Seminar. By Professor Jordan: Analytical mechanics. By Professor Wheeler: Higher algebra.

Massachusetts Institute of Technology, first period, June 14 to July 16. Calculus and differential equations, covering the prescribed work of the first two

years. Second period, July 27 to September 7. Courses in the first period repeated. August 8 to September 10. Courses in algebra, solid geometry and trigonometry, in preparation for fall entrance examinations in those subjects. June 14 to July 5. Course in theoretical aeronautics. July 6 to July 26. Course in aeronautics continued. June 14 to July 26. Advanced calculus.

University of Michigan, June 27 to August 19. In addition to courses in algebra, trigonometry, analytic geometry, elementary calculus, statistics, and finance, the following advanced courses will be offered: By Professor C. J. Coe: Topics in calculus. By Professor Norman H. Anning: Differential equations; Solid analytic geometry. By Professor W. O. Menge: Higher algebra. By Professor C. C. Craig: Theory of probability. By Professor W. D. Baten: Finite differences. By Professor Peter Field: Analytic projective geometry; Vector analysis. By Professor L. A. Hopkins: Analytic mechanics. By Professor W. B. Ford: Advanced Calculus; Infinite series, with special reference to Fourier Series. By Professor L. C. Karpinski: Teaching of algebra; History of geometry. By Mr. Dushnik: Empirical formulas. By Professor R. L. Wilder: Topology; Introduction to the foundations of mathematics. By Professor T. H. Hildebrandt: Theory of functions of a complex variable; Calculus of variations. By Professor G. Y. Rainich: Algebraic theory; Differential geometry. By Professor H. C. Carver: Mathematical theory of statistics. By Professor V. C. Poor: Applied mathematics—engineering problems. By Hildebrandt, Rainich, and others: Seminar in pure mathematics.

University of Minnesota, first term, June 15 to July 23. In addition to the usual elementary work the following courses will be offered: By Professor Dunham Jackson: History of ancient and modern mathematics; Frequency curves and surfaces. By Assistant Professor Gladys Gibbens: Differential equations. By Professors Jackson, Underhill, and Gibbens: Reading in advanced mathematics. Second term, July 25 to August 27. By Associate Professor Anthony L. Underhill: Elliptic integrals. By Professors Underhill and Carlson: Reading in advanced mathematics.

Northwestern University, June 20 to August 13. In addition to courses in trigonometry, college algebra, analytic geometry, differential and integral calculus, the following advanced courses will be offered: By Dr. H. L. Garabedian: Introduction to modern geometry. By Professor D. R. Curtiss: Advanced Calculus. By Professor H. A. Simmons: The theory of numbers.

University of Pennsylvania, July 5 to August 13. In addition to the elementary courses, the following advanced courses will be offered: By Professor G. H. Hallett: Infinite series. By Professor M. J. Babb: Differential equations. By Professor D. N. Lehmer of the University of California: Theory of numbers; Modern analytic geometry. By Professor J. R. Kline: Theory of functions of a complex variable. By Professor J. A. Shohat: Definite integrals.

University of Pittsburgh, June 13 to August 19. In addition to the undergraduate courses the following more advanced courses will be offered: By Professor K. D. Swartzel: Teaching of mathematics; Functions of a complex va-

riable. By Professor F. A. Foraker: Modern synthetic geometry; Projective geometry. By Professor J. S. Taylor: Advanced calculus; Functions of a real variable. By Associate Professor Culver: Differential equations; Theory of equations, or Theory of groups. Other graduate courses will be offered should demand warrant.

Syracuse University, June 27 to August 5. In addition to the regular courses in plane and solid geometry, intermediate and advanced algebra, trigonometry, analytic geometry, and differential and integral calculus, the following courses will be offered: By Professor May N. Harwood: Differential equations. By Professor I. S. Carroll: Teaching of mathematics. By Professor F. F. Decker: Introduction to theory of groups or History of Mathematics. By Professor A. D. Campbell: Functions of a real variable or Vector Analysis.

University of Texas, first term, June 7 to July 18; second term, July 20 to August 29. In addition to freshman courses the first term, the following courses are offered: First term. By Professor R. L. Moore: Foundations of geometry; Theory of sets. By Professor E. L. Dodd: Mathematical statistics; Analytic Functions. By Professor H. J. Ettlinger: Ruler and compass constructions; Fourier series. By Professor H. S. Vandiver: Number theory; Finite groups. By Professor P. M. Batchelder: Teaching problems in mathematics; Difference equations. By Professor Mary E. Decherd: Calculus. By Professors C. M. Cleveland and H. V. Craig: Advanced calculus. Second term. By Professor A. E. Cooper: Advanced calculus; Continuous groups. By Professor R. N. Haskell: Calculus; Potential theory. By Mr. E. C. Klipple: Differential equations; Non-Euclidean geometry.

University of Vermont. The following courses are offered by Professors Bullard, Butterfield, Millington: Courses in algebra, plane trigonometry, solid geometry, differential and integral calculus, differential equations, astronomy and history of mathematics.

University of Wisconsin, six weeks session, June 27 to August 5. By Professor T. Bennett: Differential equations; College geometry. By Dr. M. L. Hartung: The content of secondary mathematics; Teaching of mathematics. By Professor H. W. March: Advanced calculus. By Professor E. B. Skinner: Theory of equations; Finite groups. By Professor I. S. Sokolnikoff: Differential calculus; Vector analysis. Special nine weeks session for graduates, June 27 to August 26. By Professor R. E. Langer: The Lie theory of differential equations; Fourier's series. By Professor H. W. March: Theory of potential.

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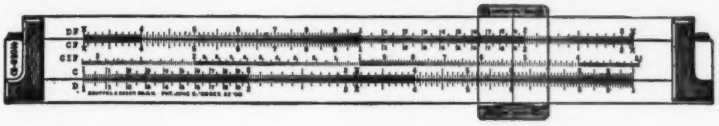

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DIRECTORY

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CHANGE OF ADDRESS: Members should send notice of any change of address to the SECRETARY-TREASURER, W. D. CAIRNS, Oberlin, Ohio, before the 10th of each month.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Sixteenth Summer Meeting of the Association, Los Angeles, Calif., Aug. 29-30, 1932.

The following is a list of the sections of the Association, with dates of those section meetings which have been scheduled for 1932 and reported to the Secretary.

ILLINOIS, Urbana, May 6-7.
INDIANA, Indianapolis, May 6-7.
IOWA, Cedar Falls, April 29-30.
KANSAS.
KENTUCKY.
LOUISIANA-MISSISSIPPI, Oxford, Miss., March 11-12.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, College Park, Md., May.
MICHIGAN, Ann Arbor, March 19.
MINNESOTA.
MISSOURI.
NEBRASKA.
OHIO, Columbus, Ohio, April 7.
PHILADELPHIA, Philadelphia, Pa., Nov. 26.
ROCKY MOUNTAIN, Laramie, Wyo., April 15-16.
SOUTHEASTERN, Gainesville, Fla., Mar. 18-19.
SOUTHERN CALIFORNIA, San Diego, March 26.
TEXAS, Austin, Jan. 30.

AFFILIATED ORGANIZATIONS: THE NEW ENGLAND ASSOCIATION OF TEACHERS OF MATHEMATICS. THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS.

NEW MATHEMATICAL PERIODICALS

By RAYMOND CLARE ARCHIBALD, Brown University

In this MONTHLY last October (p. 436–439) I gave a list of 32 periodicals which had been started during the five years 1926–1930 and which contained mathematical material of importance. Eight more periodicals are now to be added to this list, namely three from Japan, two from Germany, one from India, one from Italy, and one from Russia. The distribution of the 40 new periodicals is then as follows: Roumania (7), Italy (6), Germany (4), Japan (4), Poland (4), Argentina (3), United States (3), India (2), Russia (2), Czechoslovakia (1), England (1), Holland (1), Hungary (1), and Switzerland (1). Since seven of the eight periodicals now mentioned are research publications, at least 23 of the 40 may be so characterized. The titles of the new periodicals and additional notes concerning five of those already listed are given below.

2. *Bulletin de Mathématiques et de Physiques, Pures et Appliquées de l'Ecole Polytechnique de Bucarest*, v. 2, 1930–31, 246 p.

4. *Mathematica*, Kolozsvar [Cluj], v. 3, 1930, 157 p.; v. 4, 1930, 16+196 p.

   Volume 3 is devoted to the papers and proceedings of the first Rumanian congress of mathematics held at Cluj, 9–12 May 1929. Although dated 1930 it was really not published till 1931, after volume 4. Many of the articles in *Mathematica* have been previously published in *Societatea de Stiinte din Cluj, Buletinul*.

14. *Mathesis Polska*, Warsaw, v. 5, 1930, 8+218 p.; v. 6, nos. 1–8, January–October 1931, p. 1–170.

    As a supplement of each number of volume 6 appears a number of *Uranja*, volume 10, journal of the Polish Society of Friends of Astronomy.

16. *Studia Mathematica*, Lemberg, v. 2, 1930, 4+251 p.; v. 3, 1931, 4+247 p.

31. *Journal de la Société Physico-Mathématique de Leningrade.*

    According to an article published in Berlin the Soviet government abolished further publication of this journal after volume 2.

33. *Blätter für Versicherungs-Mathematik und verwandte Gebiete*, (published by the Deutscher Verein für Versicherungs-Wissenschaft, Berlin, as supplement to its *Zeitschrift für die gesammte Versicherungs-Wissenschaften*), v. 1, July 1928–October 1930, 4+445 p.; v. 2, parts 1–4, 1931, p. 1–167.

    Quarterly, in German, ten numbers to a volume. Contains well-edited articles, obituary notices, and reviews of current literature. A number of the articles are of considerable mathematical interest. In the issue for April 1931 there is a German-English, English-German dictionary of technical terms used in life insurance.

34. *Ingenieur-Archiv*, Berlin, v. 1, December 1929–December 1930, 4+648 p.; v. 2, March 1931–February 1932, 4+674 p.

    A periodical with many articles which the all-round mathematician cannot afford to neglect.